Universal connection and curvature for statistical manifold geometry

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Abstract

Statistical manifolds are representations of smooth families of probability density functions that allow differential geometric methods to be applied to problems in stochastic processes, mathematical statistics and information theory. It is common to have to consider a number of linear connections on a given statistical manifold and so it is important to know the corresponding universal connection and curvature; then all linear connections and their curvatures are pullbacks. An important class of statistical manifolds is that arising from the exponential families and one particular family is that of gamma distributions, which we showed recently to have important uniqueness properties in stochastic processes. Here we provide formulae for universal connections and curvatures on exponential families and give an explicit example for the manifold of gamma distributions.

1 Introduction

Information geometry is the study of Riemannian geometric properties of statistical manifolds consisting of smooth families of probability density functions. Such manifolds are endowed with the information metric of Rao [22], which arose from the Fisher information matrix [13]. These parts of mathematical statistics have deep relations with general information theory; see eg Roman [23] for a modern account of information theory from a mathematical viewpoint.

For our present purposes we may view a probability density function on $\Omega \subset \mathbb{R}^m$ as a subadditive measure function of unit weight, namely, a nonnegative map

$$\begin{aligned} f &: \Omega \longrightarrow [0,\infty) \\ \int_\Omega f &= 1 \\ \int_{A\cup B} f &\leq \int_A f + \int_B f, \quad \forall A, B \subseteq \Omega. \end{aligned}$$

Usually, a probability density function depends on a set of parameters, $\theta^1, \theta^2, \ldots, \theta^n$ and we say that we have an n-dimensional family of probability density functions. Let Θ be the parameter space of an n-dimensional smooth such family defined on some fixed event space Ω

$$\{p_\theta | \theta \in \Theta\} \ \text{ with } \int_\Omega p_\theta = 1 \quad \text{for all } \theta \in \Theta.$$

Then, the derivatives of the log-likelihood function, $l = \log p_\theta$, yield a matrix with entries

$$g_{ij} = \int_\Omega p_\theta \left(\frac{\partial l}{\partial \theta^i} \frac{\partial l}{\partial \theta^j} \right) = -\int_\Omega p_\theta \left(\frac{\partial^2 l}{\partial \theta^i \partial \theta^j} \right),$$

for coordinates (θ^i) about $\theta \in \Theta \subseteq \mathbb{R}^n$.

This gives rise to a positive definite matrix, so inducing a Riemannian metric g on Θ using for coordinates the parameters (θ^i); this metric is called the information metric for the family of probability density functions—the second equality here is subject to certain regularity conditions. Amari [1] and Amari and Nagaoka [2] provide modern accounts of the differential geometry that arises from the Fisher information metric.

2 Systems of connections and universal objects

The concept of system (or structure) of connections was introduced by Mangiarotti and Modugno [18, 19], they were concerned with finite-dimensional bundle representations of the space of all connections on a fibred manifold. On each system of connections there exists a unique universal connection of which every connection in the family of connections is a pullback. A similar relation holds between the corresponding universal curvature and the curvatures of the connections of the system. This is a different representation of an object similar to that introduced by Narasimhan and Ramanan [20], [21] for G-bundles, also allowing a proof of Weil's theorem (cf. [16, 14, 8]).

Definition 2.1 ***A system of connections*** *on a fibred manifold* $p : E \longrightarrow M$ *is a fibred manifold* $p_c : C \longrightarrow M$ *together with a first jet-valued fibred morphism*

$$\xi : C \times_M E \longrightarrow JE$$

over M*, such that each section* $\tilde{\Gamma} : M \longrightarrow C$ *determines a unique connection* $\Gamma = \xi \circ (\tilde{\Gamma} \circ p, I_E)$ *on* E*. Then* C *is the* ***space of connections*** *of the system.*

In the sequel we are interested in the system of linear connections on a Riemannian manifold. The system of all linear connections is the subject of studies in eg. [16, 14, 17, 8, 9, 10, 7, 12].

Theorem 2.1 ([18, 19]) *Let* (C, ξ) *be a system of connections on a fibred manifold* $p : E \longrightarrow M$*. Then there is a unique connection form* $\Lambda : C \times_M E \longrightarrow J(C \times_M E)$ *On the fibred manifold* $\pi_1 : C \times_M E \longrightarrow C$ *with the coordinate expression*

$$\Lambda = dx^\lambda \otimes \partial_\lambda + dc^a \otimes \partial_a + \xi^i_\lambda \, dx^\lambda \otimes \partial_i.$$

This Λ *is called the* ***universal connection*** *because it describes all the connections of the system.*

Explicitly, each $\tilde{\Gamma} \in Sec(C/M)$ *gives an injection* $(\tilde{\Gamma} \circ p, I_E)$*, of* E *into* $C \times E$*, which is a section of* π_1 *and* Γ *coincides with the restriction of* Λ *to this section:*

$$\Lambda_{|(\tilde{\Gamma} \circ p, I_E)E} = \Gamma.$$

A similar relation holds between its curvature Ω*, called* ***universal curvature****, and the curvatures of the connections of the system.*

$$\Omega = \frac{1}{2} [\Lambda, \Lambda] = d_\Lambda \Lambda : C \times_M E \longrightarrow \wedge^2 (T^* C) \otimes_E V(E).$$

So the universal curvature Ω *has the coordinate expression:*

$$\Omega = \frac{1}{2} \left(\xi^j_\lambda \, \partial_j \xi^i_\eta \, dx^\lambda \wedge dx^\eta + 2 \, \partial_a \xi^i_\eta \, dx^a \wedge dx^\eta \right) \otimes \partial_i \, .$$

3 Exponential family of probability density functions on $\mathbb{R}$

An important class of statistical manifolds is that arising from the so-called exponential family [2] and one case is that of gamma distributions, which we showed recently in [4, 5] to have important uniqueness properties for near-random stochastic processes. Note also that Hwang and Hu [15] provided an important new characterization of gamma distributions, which helps understanding of their common application in modelling real processes. More details on statistical manifolds in general can be found in [1, 2] and we have provided in [6] explicit geometric neighbourhoods of independence for common bivariate processes. In the present section we shall be concerned with the system of all linear connections on the manifold of an arbitrary exponential family, using the tangent bundle or the frame bundle to give the system space. We provide formulae for the universal connections and curvatures and give an explicit example for the manifold of gamma distributions.

An n-dimensional set of probability density functions $S = \{p_\theta | \theta \in \Theta \subset \mathbb{R}^n\}$ for random variable $x \in \Omega \subseteq \mathbb{R}$ is said to be an **exponential family** [2] when the density functions can be expressed in terms of functions $\{C, F_1, ..., F_n\}$ on $\mathbb{R}$ and a function φ on Θ as:

$$p_\theta(x) = e^{\{C(x)+\sum_i(\theta^i F_i(x))-\varphi(\theta)\}}.$$

Then we say that (θ^i) are its **natural** coordinates, and φ is its **potential function**. From the normalization condition $\int_\Omega p_\theta(x)\,dx = 1$ we obtain:

$$\varphi(\theta) = \log \int_\Omega e^{\{C(x)+\sum_i(\theta^i F_i(x)\})}\,dx\,.$$

From the definition of an exponential family, and putting $\partial_i = \frac{\partial}{\partial\theta^i}$, we use the log-likelihood function $l(\theta, x) = \log(p_\theta(x))$ to obtain

$$\partial_i l(\theta, x) \quad = \quad F_i(x) - \partial_i\varphi(\theta)$$

and

$$\partial_i\partial_j l(\theta, x) \quad = \quad -\partial_i\partial_j\varphi(\theta)\,.$$

The Fisher information metric g [1, 2] on the n-dimensional space of parameters $\Theta \subset \mathbb{R}^n$, equivalently on the set $S = \{p_\theta | \theta \in \Theta \subset \mathbb{R}^n\}$, has coordinates:

$$[g_{ij}] = -\int_\Omega [\partial_i\partial_j l(\theta, x)]\; p_\theta(x)\; dx = \partial_i\partial_j\varphi(\theta) = \varphi_{ij}(\theta)\;.$$

Then, (S, g) is a Riemannian n-manifold with Levi-Civita connection given by:

$$\begin{aligned}\Gamma^k_{ij}(\theta) &= \sum_{h=1}^n \frac{1}{2}\,g^{kh}\;(\partial_i g_{jh} + \partial_j g_{ih} - \partial_h g_{ij}) \\ &= \sum_{h=1}^n \frac{1}{2}\,g^{kh}\,\partial_i\partial_j\partial_h\varphi(\theta) = \sum_{h=1}^n \frac{1}{2}\,\varphi^{kh}(\theta)\,\varphi_{ijh}(\theta)\end{aligned}$$

where $[\varphi^{hk}(\theta)]$ represents the inverse to $[\varphi_{hk}(x)]$.

Next we obtain a family of symmetric connections which includes the Levi-Civita case and has significance in mathematical statistics. Consider for $\alpha \in \mathbb{R}$ the function $\Gamma^{(\alpha)}_{ij,k}$ which maps each point $\theta \in \Theta$ to the following value:

$$\begin{aligned}\Gamma^{(\alpha)}_{ij,k}(\theta) &= \int_\Omega \left(\partial_i\partial_j l + \frac{1-\alpha}{2}\,\partial_i l\,\partial_j l\right)\partial_k l\;p_\theta \\ &= \frac{1-\alpha}{2}\,\partial_i\partial_j\partial_k\varphi(\theta) = \frac{1-\alpha}{2}\,\varphi_{ijk}(\theta)\,.\end{aligned}$$

So we have an affine connection $\nabla^{(\alpha)}$ on the statistical manifold (S, g) defined by

$$g(\nabla^{(\alpha)}_{\partial_i}\partial_j, \partial_k) \quad = \quad \Gamma^{(\alpha)}_{ij,k}\,,$$

where g is the Fisher information metric. We call this $\nabla^{(\alpha)}$ the α-connection and it is clearly a symmetric connection and defines an α-curvature. We have also

$$\begin{aligned}\nabla^{(\alpha)} &= (1-\alpha)\;\nabla^{(0)} + \alpha\,\nabla^{(1)}\,, \\ &= \frac{1+\alpha}{2}\,\nabla^{(1)} + \frac{1-\alpha}{2}\,\nabla^{(-1)}\,.\end{aligned}$$

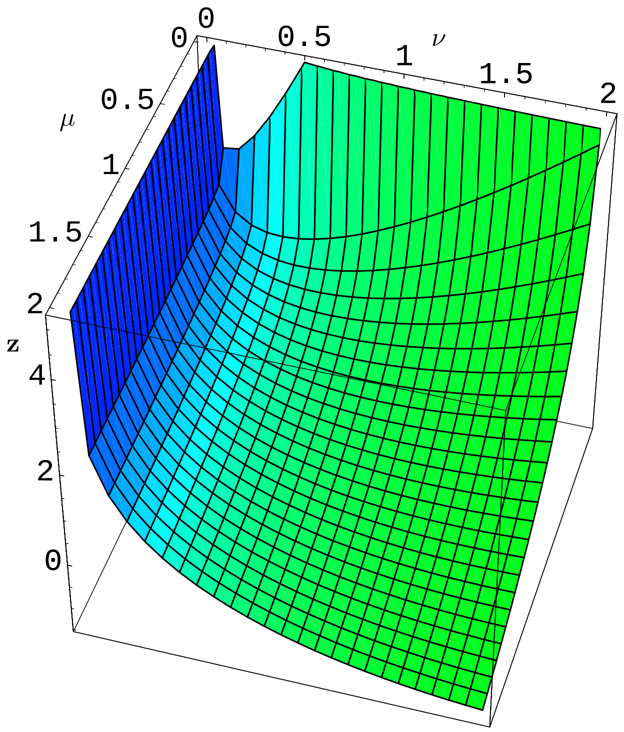

Figure 1: *Affine immersion using natural coordinates (μ, ν) in $\mathbb{R}^3$ for the gamma 2-manifold $(\mathcal{G}, g)$. The surface is shaded by the Gaussian curvature $K_{\mathcal{G}}$ which is independent of μ and monotonically decreases from $-\frac{1}{4}$ to almost $-\frac{1}{2}$ as ν increases to 2.*

For a submanifold $M \subset S$, the α-connection on M is simply the restriction with respect to g of the α-connection on S. Note that the 0-connection is the Riemannian or Levi-Civita connection with respect to the Fisher metric and its uniqueness implies that an α-connection is a metric connection if and only if $\alpha = 0$.

3.1 Example: Gamma 2-manifold $(\mathcal{G}, g)$

Gamma distributions form an exponential family with probability density functions:

$$\mathcal{G} = \{p(x; \mu, \nu) = \mu^{\nu} \, \frac{x^{\nu-1}}{\Gamma(\nu)} \, e^{-x\mu} \quad \text{for} \quad \mu, \nu \in \mathbb{R}^+\}. \tag{3.1}$$

The gamma distributions are very important in, information theory, mathematical statistics and stochastic processes. This is because they contain as a special case, $\nu = 1$, the negative exponential distribution that represents random states, ie Poisson processes, and because of certain uniqueness properties [5, 15]. It turns out that $(\theta^i) = (\mu, \nu)$ is a natural coordinate system with corresponding potential function

$$\varphi(\mu, \nu) = \log \Gamma(\nu) - \nu \, \log \mu \ . \tag{3.2}$$

The information metric g has arc length function

$$ds_g^2 \quad = \quad \frac{\nu}{\mu^2} \, d\mu^2 - \frac{2}{\mu} \, d\mu \, d\nu + \psi'(\nu) \, d\nu^2 \quad \text{for} \quad \mu, \nu \in \mathbb{R}^+$$

where $\psi(\nu) = \frac{\Gamma'(\nu)}{\Gamma(\nu)}$ is the digamma function. The independent nonzero Levi-Civita connection compo-

Univariate density	Coordinates	Mean	Variance	$R^{(\alpha)}$
Gaussian	(μ, σ)	μ	σ	$\left(\alpha^2 - 1\right)$
Gamma	(μ, ν)	μ	μ^2/ν	$\frac{\left(1-\alpha^2\right)\left(\psi'(\nu)+\nu\,\psi''(\nu)\right)}{2\,(\nu\,\psi'(\nu)-1)^2}$
Exponential	μ	μ	μ^2	0

Table 1: *α-Scalar curvature $R^{(\alpha)}$ of the univariate Gaussian, gamma and exponential statistical manifolds; the logarithmic derivative of the gamma function is denoted by $\psi = \Gamma'/\Gamma$. The case $\alpha = 0$ corresponds to the Levi-Civita connection.*

nents with respect to the natural coordinates (μ, ν) are:

$$\begin{aligned}
\Gamma^1_{11} &= \frac{(1 - 2\,\nu\,\psi'(\nu))}{2\,\mu\,(-1 + \nu\,\psi'(\nu))}\,, \\
\Gamma^1_{12} &= \frac{\psi'(\nu)}{2\,(\nu\,\psi'(\nu) - 1)}\,, \\
\Gamma^1_{22} &= \frac{\mu\,\psi''(\nu)}{2\,(\nu\,\psi'(\nu) - 1)}\,, \\
\Gamma^2_{11} &= \frac{\nu}{2\,\mu^2\,(1 - \nu\,\psi'(\nu))}\,, \\
\Gamma^2_{12} &= \frac{1}{2\,\mu\,(\nu\,\psi'(\nu) - 1)}\,, \\
\Gamma^2_{22} &= \frac{\nu\,\psi''(\nu)}{2\,(\nu\,\psi'(\nu) - 1)}.
\end{aligned}$$

The Riemannian 2-manifold $(\mathcal{G}, g)$ has been shown by Dodson and Matsuzoe [11] to admit an affine immersion in $\mathbb{R}^3$. This is depicted in Figure 1, shaded by the Gaussian curvature $K_{\mathcal{G}}$, which is independent of μ and, with increasing ν, $K_{\mathcal{G}}$ monotonically decreases from $-\frac{1}{4}$ to $-\frac{1}{2}$:

$$K_{\mathcal{G}} = \frac{\psi'(\nu) + \nu\,\psi''(\nu)}{4\,(\nu\,\psi'(\nu) - 1)^2}.$$

To compute the α-connection components it is convenient here to change to the orthogonal coordinates $(\beta = \nu/\mu, \nu)$ for which the metric components are given by

$$ds^2 = \frac{\nu}{\beta^2}\,d\beta^2 + \left(\psi'(\nu) - \frac{1}{\nu}\right)\,d\nu^2 \quad \text{for} \quad \beta, \nu \in \mathbb{R}^+\ .$$

Proposition 3.1 (Arwini [3]) *The independent nonzero components, $\Gamma^{(\alpha)i}_{jk}$, of $\nabla^{(\alpha)}$ are*

$$\begin{aligned}
\Gamma^{(\alpha)1}_{11} &= -\frac{\alpha + 1}{\beta}\,, \\
\Gamma^{(\alpha)1}_{12} &= \frac{\alpha + 1}{2\,\nu}\,, \\
\Gamma^{(\alpha)2}_{11} &= \frac{(\alpha - 1)\,\nu}{2\,\beta^2\,(\nu\,\psi'(\nu) - 1)}\,, \\
\Gamma^{(\alpha)2}_{22} &= \frac{(1 - \alpha)\,\left(1 + \nu^2\,\psi''(\nu)\right)}{2\,\nu\,(\nu\,\psi'(\nu) - 1)}\,.
\end{aligned}$$

□

Corollary 3.1 *The Levi Civita connection of $(\mathcal{G}, g)$ is recovered by $\Gamma^{(0)i}_{jk}$ in (β, ν) coordinates and then the curves $\nu =$ constant are geodesics.* □

Bivariate density	**Coordinates**	**Covariance**	$R^{(\alpha)}$
Mckay M	(α_1, c, α_2)	α_1/c^2	$R_M^{(\alpha)}$
$M_1 \subset M$: $\alpha_1 = 1$	(c, α_2)	$1/c^2$	${R_M}_1^{(\alpha)}$
$M_2 \subset M$: $\alpha_2 = 1$	(α_1, c)	α_1/c^2	${R_M}_2^{(\alpha)}$
M_3: $\alpha_1 + \alpha_2 = 1$	(α_1, c)	α_1/c^2	0

Table 2: *α-Scalar curvature $R^{(\alpha)}$ of the McKay bivariate gamma manifold; see § 3.2 for the formulae $R_M^{(\alpha)}$, ${R_M}_1^{(\alpha)}$, ${R_M}_2^{(\alpha)}$. The case $\alpha = 0$ corresponds to the Levi-Civita connection.*

Bivariate density	**Coordinates**	**Covariance**	$R^{(\alpha)}$
Freund F	$(\alpha_1, \beta_1, \alpha_2, \beta_2)$	$\frac{\beta_1\,\beta_2 - \alpha_1\,\alpha_2}{\beta_1\,\beta_2\,(\alpha_1+\alpha_2)^2}$	$\frac{-3\left(\alpha^2-1\right)}{2}$
$F_1 \subset F$: $\beta_i = \alpha_i$	(α_1, α_2)	0	0
$F_2 : \alpha_1 = \alpha_2, \beta_1 = \beta_2$	(α_1, β_1)	$\frac{1}{4}\left(\frac{1}{{\alpha_1}^2} - \frac{1}{{\beta_1}^2}\right)$	0
F_3: $\beta_i = \alpha_1 + \alpha_2$	$(\alpha_1, \alpha_2, \beta_2)$	$\frac{{\alpha_1}^2+\alpha_1\,\alpha_2+{\alpha_2}^2}{(\alpha_1+\alpha_2)^4}$	0

Table 3: *α-Scalar curvature $R^{(\alpha)}$ of the Freund bivariate exponential manifold. The case $\alpha = 0$ corresponds to the Levi-Civita connection.*

Bivariate density	**Coordinates**	**Covariance**	$R^{(\alpha)}$
Gaussian N	$(\mu_1, \mu_2, \sigma_1, \sigma_{12}, \sigma_2)$	σ_{12}	$\frac{9\left(\alpha^2-1\right)}{2}$
$N_1 \subset N$: $\sigma_{12} = 0$	$(\mu_1, \mu_2, \sigma_1, \sigma_2)$	0	$2\left(\alpha^2 - 1\right)$
N_2: $\sigma_i = \sigma, \mu_i = \mu$	$(\mu, \sigma, \sigma_{12})$	σ_{12}	$\left(\alpha^2 - 1\right)$
N_3: $\mu_1 = \mu_2 = 0$	$(\sigma_1, \sigma_2, \sigma_{12})$	σ_{12}	$2\left(\alpha^2 - 1\right)$

Table 4: *α-Scalar curvature $R^{(\alpha)}$ of the bivariate Gaussian manifold. The case $\alpha = 0$ corresponds to the Levi-Civita connection.*

3.2 α-Scalar curvature for common distributions

For convenience of reference we summarize curvature results in Table 1 for univariate Gaussian, gamma and exponential distributions and in Tables 2,3,4 respectively for bivariate gamma, exponential and Gaussian distributions, from recent work of Arwini and Dodson [4, 5, 6]. We have used *Mathematica* for many calculations and we can make available the associated interactive Notebooks. The α-scalar curvature for the McKay bivariate gamma manifold M and its submanifolds M_1, M_2, have long expressions so we give them here:

$$\begin{aligned}
R_M^{(\alpha)} &= \left(1-\alpha^2\right)\Big(\frac{\psi'(\alpha_2)\left(\psi'(\alpha_1)\left(\psi'(\alpha_1)+\psi'(\alpha_2)\right)-2\psi''(\alpha_1)\right)-2\psi'(\alpha_1)\psi''(\alpha_2)}{2(\psi'(\alpha_1)+\psi'(\alpha_2)-\psi'(\alpha_1)\psi'(\alpha_2)\left(\alpha_1+\alpha_2\right))^2} \\
&\quad + \frac{\left(\psi'(\alpha_2)^2\psi''(\alpha_1)+\left(\psi'(\alpha_1)^2-\psi''(\alpha_1)\right)\psi''(\alpha_2)\right)\left(\alpha_1+\alpha_2\right)}{2(\psi'(\alpha_1)+\psi'(\alpha_2)-\psi'(\alpha_1)\psi'(\alpha_2)\left(\alpha_1+\alpha_2\right))^2}\Big), \\
&\quad \psi(\alpha_i) = \frac{\Gamma'(\alpha_i)}{\Gamma(\alpha_i)}. \\
R_{M_1}^{(\alpha)} &= \frac{\left(1-\alpha^2\right)\left(\psi'(\alpha_2)+\psi''(\alpha_2)\left(1+\alpha_2\right)\right)}{2(\psi'(\alpha_2)\left(1+\alpha_2\right)-1)^2} \\
R_{M_2}^{(\alpha)} &= \frac{\left(1-\alpha^2\right)\left(\psi'(\alpha_1)+\psi''(\alpha_1)\left(1+\alpha_1\right)\right)}{2(\psi'(\alpha_1)\left(1+\alpha_1\right)-1)^2}.
\end{aligned}$$

4 Systems of linear connections

4.1 Tangent bundle system: $C_T \times TM \longrightarrow JTM$

The system of all linear connections on a manifold M has a representation on the tangent bundle

$$E = TM \longrightarrow M$$

with system space

$$C_T = \{\alpha \otimes j\gamma \in T^*M \otimes_M JTM \mid j\gamma : TM \longrightarrow TTM \text{ projects onto } I_{TM}\}$$

Here we view I_{TM} as a section of $T^*M \otimes TM$, which is a subbundle of $T^*M \otimes TTM$, with local expression $dx^\lambda \otimes \partial_\lambda$.

The fibred morphism for this system is given by

$$\begin{array}{rcl} \xi_T : C_T \times_M TM & \longrightarrow & JTM \subset T^*M \otimes_{TM} TTM\,, \\ (\alpha \otimes j\gamma, \nu) & \longmapsto & \alpha(\nu) j\gamma. \end{array}$$

In coordinates (x^λ) on M and (y^λ) on TM

$$\begin{array}{rcl} \xi_T & = & dx^\lambda \otimes (\partial_\lambda - \gamma^i_\lambda\, \partial_i) \\ & = & dx^\lambda \otimes (\partial_\lambda - y^j\, \Gamma^i_{j\lambda}\, \partial_i) \\ & = & dx^\lambda \otimes (\partial_\lambda - y^j\, (\sum_{h=1}^{n} \frac{1}{2}\, \varphi^{ih}\, \varphi_{j\lambda h})\, \partial_i) \end{array}$$

Each section of $C_T \longrightarrow M$, such as $\tilde{\Gamma} : M \longrightarrow C_T : (x^\lambda) \longrightarrow (x^\lambda, \gamma_{\eta\theta})$; determines the unique linear connection $\Gamma = \xi_T \circ (\tilde{\Gamma} \circ \pi_T, I_{TM})$ with Christoffel symbols $\Gamma^\lambda_{\eta\theta}$.

On the fibred manifold $\pi_1 : C_T \times_M TM \longrightarrow C_T$; the universal connection is given by:

$$\begin{array}{rcl} \Lambda_T : C_T \times_M TM & \longrightarrow & J(C_T \times_M TM) \subset T^*C_T \otimes T(C_T \times_M TM)\,, \\ (x^\lambda, v^\lambda_{\eta\kappa}, y^\lambda) & \longmapsto & [(X^\lambda, V^\lambda_{\eta\kappa}) \longrightarrow (X^\lambda, V^\lambda_{\eta\kappa}, Y^\eta V^\lambda_{\eta\kappa} X^\kappa)]. \end{array}$$

briefly,

$$\begin{array}{rcl} \Lambda_T & = & dx^\lambda \otimes \partial_\lambda + dv^a \otimes \partial_a + y^\eta v^i_{\eta\kappa}\, dx^\kappa \otimes \partial_i \\ & = & dx^\lambda \otimes \partial_\lambda + dv^a \otimes \partial_a + y^\eta\, (\sum_{h=1}^{n} \frac{1}{2}\, \varphi^{ih}\, \varphi_{\eta\kappa h})\, dx^\kappa \otimes \partial_i. \end{array}$$

Explicitly, each $\tilde{\Gamma} \in Sec(C_T/M)$ gives an injection $(\tilde{\Gamma} \circ \pi_T, I_{TM})$, of TM into $C_T \times TM$, which is a section of π_1 and Γ coincides with the restriction of Λ_T to this section:

$$\Lambda_{T|(\tilde{\Gamma} \circ \pi_T, I_{TM})TM} = \Gamma.$$

and the universal curvature of the connection Λ is given by:

$$\Omega_T = d_{\Lambda_T} \Lambda_T : C_T \times_M TM \longrightarrow \wedge^2 (T^*C_T) \otimes_{TM} V(TM).$$

So here the universal curvature Ω_T has the coordinate expression:

$$\begin{array}{rcl} \Omega_T & = & \frac{1}{2} \left(y^k v^j_{k\lambda}\, \partial_j y^m v^i_{m\eta}\, dx^\lambda \wedge dx^\eta + 2\, \partial_a y^m v^i_{m\eta}\, dx^a \wedge dx^\eta \right) \otimes \partial_i \\ & = & \frac{1}{2} \left(y^k (\sum_{h=1}^{n} \frac{1}{2}\, \varphi^{jh}\, \varphi_{k\lambda h}) \partial_j y^m (\sum_{h=1}^{n} \frac{1}{2}\, \varphi^{ih}\, \varphi_{m\eta h}) dx^\lambda \wedge dx^\eta \right) \otimes \partial_i \\ & & + \left(\partial_a y^m (\sum_{h=1}^{n} \frac{1}{2}\, \varphi^{ih}\, \varphi_{m\eta h}) dx^a \wedge dx^\eta \right) \otimes \partial_i\,. \end{array}$$

4.2 Frame bundle system: $C_F \times FM \longrightarrow JFM$

A linear connection is also a principal (i.e. group invariant) connection on the principal bundle of frames FM with:

$$E = FM \longrightarrow M = FM/G$$

consisting of linear frames (ordered bases for tangent spaces) with structure group the general linear group, $G = Gl(n)$. Here the system space is

$$C_F = JFM/G \subset T^*M \otimes_{TM} TFM/G,$$

consisting of G-invariant jets. The system morphism is

$$\begin{array}{rcl} \xi_F : C_F \times FM & \longrightarrow & JFM \subset T^*M \otimes_{TM} TFM\,, \\ ([js_x], b) & \longmapsto & [T_xM \longmapsto T_bFM]. \end{array}$$

In coordinates

$$\begin{array}{rcl} \xi_F & = & dx^\lambda \otimes (\partial_\lambda - X^\eta\, \partial_\eta s^\lambda_\kappa)\, \tilde{\partial}_{\kappa^\lambda} \\ & = & dx^\lambda \otimes (\partial_\lambda - X^\eta\, \Gamma^\lambda_{\eta\kappa})\, \tilde{\partial}_{\kappa^\lambda} \\ & = & dx^\lambda \otimes (\partial_\lambda - X^\eta \displaystyle\sum_{h=1}^{n} \frac{1}{2}\, \varphi^{\lambda h}\, \varphi_{\eta\kappa h})\, \tilde{\partial}_{\kappa^\lambda} \end{array}$$

where $\tilde{\partial}_{\kappa^\lambda} = \frac{\partial}{\partial b^\lambda_\kappa}$ is the natural base on the vertical fibre of T_bFM induced by coordinates (b^λ_κ) on FM. Each section of $C_F \longrightarrow M$ that is projectable onto I_{TM}, such as, $\hat{\Gamma} : M \longrightarrow C_F : (x^\lambda) \longrightarrow (x^\lambda, [j\gamma_x])$ with $\Gamma^\lambda_{\eta\kappa} = \partial_\eta s^\lambda_\kappa$; determines the unique linear connection $\Gamma = \xi_F \circ (\hat{\Gamma} \circ \pi_F, I_{FM})$ with Christoffel symbols $\Gamma^\lambda_{\eta\kappa}$. On the principal G-bundle $\pi_1 : C_F \times_M FM \longrightarrow C_F$ the universal connection is given by:

$$\begin{array}{rcl} \Lambda_F : C_F \times_M FM & \longrightarrow & J(C_F \times_M FM) \subset T^*C_F \otimes_{FM} T(C_F \times_{FM} FM)\,, \\ (x^\lambda, v^\lambda_{\eta\kappa}, b^\eta_\kappa) & \longmapsto & [(X^\lambda, Y^\lambda_{\eta\kappa}) \longrightarrow (X^\lambda, Y^\lambda_{\eta\kappa}, b^\eta_\kappa v^\lambda_{\eta\theta} X^\theta)]. \end{array}$$

Briefly,

$$\begin{array}{rcl} \Lambda_F & = & dx^\lambda \otimes \partial_\lambda + dv^a \otimes \partial_a + b^\eta_\kappa v^\lambda_{\eta\theta}\, dx^\theta \otimes \tilde{\partial}_{\kappa^\lambda} \\ & = & dx^\lambda \otimes \partial_\lambda + dv^a \otimes \partial_a + b^\eta_\kappa\, (\displaystyle\sum_{h=1}^{n} \frac{1}{2}\, \varphi^{\lambda h}\, \varphi_{\eta\theta h})\, dx^\theta \otimes \tilde{\partial}_{\kappa^\lambda}. \end{array}$$

Explicitly, each $\tilde{\Gamma} \in Sec(C_F/M)$ gives an injection $(\tilde{\Gamma} \circ \pi_F, I_{FM})$, of FM into $C_F \times FM$, which is a section of π_1 and Γ coincides with the restriction of Λ_F to this section:

$$\Lambda_{F\,|(\tilde{\Gamma}\circ\pi_F, I_{FM})FM} = \Gamma$$

and the universal curvature of the connection Λ is given by:

$$\Omega = d_{\Lambda_F} \Lambda_F : C_F \times_M FM \longrightarrow \wedge^2(T^*C_F) \otimes_{FM} V(FM).$$

So here the universal curvature form Ω_F has the coordinate expression:

$$\begin{array}{rcl} \Omega_F & = & \dfrac{1}{2} \left(b^k_\kappa v^\beta_{k\lambda}\, \tilde{\partial}_{\kappa^\beta} b^m_\omega v^\alpha_{m\eta}\, dx^\lambda \wedge dx^\eta + 2\, \partial_a b^m_\omega v^\alpha_{m\eta}\, dx^a \wedge dx^\eta \right) \otimes \tilde{\partial}_{\omega^\alpha} \\ & = & \dfrac{1}{2} \left(b^k_\kappa (\displaystyle\sum_{h=1}^{n} \frac{1}{2}\, \varphi^{\beta h}\, \varphi_{k\lambda h}) \tilde{\partial}_{\kappa^\beta} b^m_\omega (\displaystyle\sum_{h=1}^{n} \frac{1}{2}\, \varphi^{\alpha h}\, \varphi_{m\eta h}) dx^\lambda \wedge dx^\eta \right) \otimes \tilde{\partial}_{\omega^\alpha} \\ & & + \left(\partial_a b^m_\omega (\displaystyle\sum_{h=1}^{n} \frac{1}{2}\, \varphi^{\alpha h}\, \varphi_{m\eta h}) dx^a \wedge dx^\eta \right) \otimes \tilde{\partial}_{\omega^\alpha}\,. \end{array}$$

5 Universal connection and curvature on the gamma manifold

For the gamma 2-manifold $(\mathcal{G}, g)$ we give explicit forms for the system space and its universal connection and curvature.

5.1 Tangent bundle system on $(\mathcal{G}, g)$

The system space is

$$C_T = \{\alpha \otimes j\gamma \in T^*\mathcal{G} \otimes_{\mathcal{G}} JT\mathcal{G} \mid j\gamma : T\mathcal{G} \longrightarrow TT\mathcal{G} \ \text{projects onto} \ I_{T\mathcal{G}}\}$$

and the system morphism is

$$\begin{aligned}
\xi_T &= dx^\lambda \otimes (\partial_\lambda - y^j \Gamma^i_{j\lambda}\, \partial_i) \\
&= dx^1 \otimes \left(\partial_1 - (\frac{(1 - 2\,\nu\,\psi'(\nu))}{2\,\mu\,(-1 + \nu\,\psi'(\nu))}\, y^1 + \frac{\psi'(\nu)}{2\,(\nu\,\psi'(\nu) - 1)}\, y^2)\, \partial_1\right) \\
&+ dx^1 \otimes \left(\partial_1 - (\frac{\nu}{2\,\mu^2\,(1 - \nu\,\psi'(\nu))}\, y^1 + \frac{1}{2\,\mu\,(\nu\,\psi'(\nu) - 1)}\, y^2)\, \partial_2\right) \\
&+ dx^2 \otimes \left(\frac{\psi'(\nu)}{2\,(\nu\,\psi'(\nu) - 1)}\, y^1 + \frac{\mu\,\psi''(\nu)}{2\,(\nu\,\psi'(\nu) - 1)}\, y^2)\, \partial_1\right) \\
&+ dx^2 \otimes \left(\partial_2 - (\frac{1}{2\,\mu\,(\nu\,\psi'(\nu) - 1)}\, y^1 + \frac{\nu\,\psi''(\nu)}{2\,(\nu\,\psi'(\nu) - 1)}\, y^2)\, \partial_2\right).
\end{aligned}$$

The universal connection on the gamma manifold is given by:

$$\begin{aligned}
\Lambda_T &= dx^\lambda \otimes \partial_\lambda + d^i_{\lambda j} \otimes \partial^i_{\lambda j} + y^\tau\, \Gamma^i_{\tau\kappa} dx^\kappa \otimes \partial_i \\
&= dx^\lambda \otimes \partial_\lambda + d^i_{\lambda j} \otimes \partial^i_{\lambda j} \\
&+ \left(\frac{(1 - 2\,\nu\,\psi'(\nu))}{2\,\mu\,(-1 + \nu\,\psi'(\nu))}\, y^1 + \frac{\psi'(\nu)}{2\,(\nu\,\psi'(\nu) - 1)}\, y^2\right) dx^1 \otimes \partial_1 \\
&+ \left(\frac{\nu}{2\,\mu^2\,(1 - \nu\,\psi'(\nu))}\, y^1 + \frac{1}{2\,\mu\,(\nu\,\psi'(\nu) - 1)}\, y^2\right) dx^1 \otimes \partial_2 \\
&+ \left(\frac{\psi'(\nu)}{2\,(\nu\,\psi'(\nu) - 1)}\, y^1 + \frac{\mu\,\psi''(\nu)}{2\,(\nu\,\psi'(\nu) - 1)}\, y^2\right) dx^2 \otimes \partial_1 \\
&+ \left(\frac{1}{2\,\mu\,(\nu\,\psi'(\nu) - 1)}\, y^1 + \frac{\nu\,\psi''(\nu)}{2\,(\nu\,\psi'(\nu) - 1)}\, y^2\right) dx^2 \otimes \partial_2\,.
\end{aligned}$$

The universal curvature on the gamma manifold is:

$$\Omega_T = \frac{1}{2}\left(y^k \Gamma^j_{k\lambda}\, \partial_j y^m \Gamma^i_{m\kappa}\, dx^\lambda \wedge dx^\kappa + 2\, \partial_a y^m \Gamma^i_{m\kappa}\, dx^a \wedge dx^\kappa\right) \otimes \partial_i \quad (i = 1, 2).$$

The analytic form of this is known [3] but is omitted here.

5.2 Frame bundle system on $(\mathcal{G}, g)$

The system space is $C_F = JF\mathcal{G}/G$ and the system morphism is

$$\begin{aligned}
\xi_F &= dx^\lambda \otimes (\partial_\lambda - X^\tau\, \Gamma^\lambda_{\tau\kappa})\, \tilde{\partial}_{\kappa^\lambda} \\
&= dx^1 \otimes \left(\partial_1 - (\frac{(1 - 2\,\nu\,\psi'(\nu))}{2\,\mu\,(-1 + \nu\,\psi'(\nu))}\, X^1 + \frac{\psi'(\nu)}{2\,(\nu\,\psi'(\nu) - 1)}\, X^2)\right) \tilde{\partial}_{1^1} \\
&+ dx^2 \otimes \left(\partial_2 - (\frac{\nu}{2\,\mu^2\,(1 - \nu\,\psi'(\nu))}\, X^1 + \frac{1}{2\,\mu\,(\nu\,\psi'(\nu) - 1)}\, X^2)\right) \tilde{\partial}_{1^2} \\
&+ dx^1 \otimes \left(\partial_1 - (\frac{\psi'(\nu)}{2\,(\nu\,\psi'(\nu) - 1)}\, X^1 + \frac{\mu\,\psi''(\nu)}{2\,(\nu\,\psi'(\nu) - 1)}\, X^2)\right) \tilde{\partial}_{2^1} \\
&+ dx^2 \otimes \left(\partial_2 - (\frac{1}{2\,\mu\,(\nu\,\psi'(\nu) - 1)}\, X^1 + \frac{\nu\,\psi''(\nu)}{2\,(\nu\,\psi'(\nu) - 1)}\, X^2)\right) \tilde{\partial}_{2^2}.
\end{aligned}$$

The universal connection on the gamma manifold is:

$$
\begin{aligned}
\Lambda_F &= dx^\lambda \otimes \partial_\lambda + d^i_{\lambda j} \otimes \partial^i_{\lambda j} + b^\kappa_\tau \, \Gamma^\lambda_{\tau\theta} dx^\theta \otimes \tilde{\partial}_{\kappa^\lambda} \\
&= dx^\lambda \otimes \partial_\lambda + d^i_{\lambda j} \otimes \partial^i_{\lambda j} \\
&+ \left(\frac{(1 - 2\,\nu\,\psi'(\nu))}{2\,\mu\,(-1+\nu\,\psi'(\nu))}\, b^1_1 + \frac{\psi'(\nu)}{2\,(\nu\,\psi'(\nu)-1)}\, b^2_1 \right) dx^1 \otimes \tilde{\partial}_{1^1} \\
&+ \left(\frac{\psi'(\nu)}{2\,(\nu\,\psi'(\nu)-1)}\, b^1_1 + \frac{\mu\,\psi''(\nu)}{2\,(\nu\,\psi'(\nu)-1)}\, b^2_1 \right) dx^2 \otimes \tilde{\partial}_{1^1} \\
&+ \left(\frac{(1 - 2\,\nu\,\psi'(\nu))}{2\,\mu\,(-1+\nu\,\psi'(\nu))}\, b^1_2 + \frac{\psi'(\nu)}{2\,(\nu\,\psi'(\nu)-1)}\, b^2_2 \right) dx^1 \otimes \tilde{\partial}_{2^1} \\
&+ \left(\frac{\psi'(\nu)}{2\,(\nu\,\psi'(\nu)-1)}\, b^1_2 + \frac{\mu\,\psi''(\nu)}{2\,(\nu\,\psi'(\nu)-1)}\, b^2_2 \right) dx^2 \otimes \tilde{\partial}_{2^1} \\
&+ \left(\frac{\nu}{2\,\mu^2\,(1-\nu\,\psi'(\nu))}\, b^1_1 + \frac{1}{2\,\mu\,(\nu\,\psi'(\nu)-1)}\, b^2_1 \right) dx^1 \otimes \tilde{\partial}_{1^2} \\
&+ \left(\frac{1}{2\,\mu\,(\nu\,\psi'(\nu)-1)}\, b^1_1 + \frac{\nu\,\psi''(\nu)}{2\,(\nu\,\psi'(\nu)-1)}\, b^2_1 \right) dx^2 \otimes \tilde{\partial}_{1^2} \\
&+ \left(\frac{\nu}{2\,\mu^2\,(1-\nu\,\psi'(\nu))}\, b^1_2 + \frac{1}{2\,\mu\,(\nu\,\psi'(\nu)-1)}\, b^2_2 \right) dx^1 \otimes \tilde{\partial}_{2^2} \\
&+ \left(\frac{1}{2\,\mu\,(\nu\,\psi'(\nu)-1)}\, b^1_2 + \frac{\nu\,\psi''(\nu)}{2\,(\nu\,\psi'(\nu)-1)}\, b^2_2 \right) dx^2 \otimes \tilde{\partial}_{2^2} \,.
\end{aligned}
$$

The universal curvature on the gamma manifold is:

$$
\Omega_F = \frac{1}{2} \left(b^k_\kappa \Gamma^\eta_{k\lambda}\, \tilde{\partial}_{\kappa^\eta} b^m_\omega \Gamma^\alpha_{m\kappa}\, dx^\lambda \wedge dx^\kappa + 2\, \partial_a b^m_\omega \Gamma^\alpha_{m\kappa}\, dx^a \wedge dx^\kappa \right) \otimes \tilde{\partial}_{\omega^\alpha}.
$$

The analytic form of this is known [3] but is omitted here.

6 Universal connection and curvature for α-connections [10]

Consider an exponential family having statistical n-manifold (M, g) and the system

$$
C \times FM \longrightarrow JFM : (\alpha, b) \mapsto \Gamma^{(\alpha)}(b)
$$

where $C = M \times \mathbb{R}$ is the direct product manifold of M with the standard real line. So the system space C consists of a stack of copies of M. Then every $\tilde{\Gamma} \in Sec(C/M)$ is a constant real function on M, so defining precisely one α-connection.

In the case of the frame bundle system, $(M \times \mathbb{R}) \times_M FM \longrightarrow JFM$, the universal connection on the system of α-connections is

$$
\begin{aligned}
\Lambda \quad &: \quad (M \times \mathbb{R}) \times_M FM \longrightarrow \\
& \qquad J((M \times \mathbb{R}) \times_M FM) \subset T^*(M \times \mathbb{R}) \otimes_{FM} T((M \times \mathbb{R}) \times_{FM} FM)\,, \\
(x^\lambda, \alpha, b^\eta_\kappa) &\longmapsto \quad [(X^\lambda, Y^\lambda_{\eta\kappa}) \longrightarrow (X^\lambda, Y^\lambda_{\eta\kappa}, b^\eta_\kappa\, \alpha\, X^\theta)].
\end{aligned}
$$

briefly,

$$
\Lambda = dx^\lambda \otimes \partial_\lambda + d\alpha \otimes \partial_\alpha + b^\eta_\kappa\, \alpha\, dx^\theta \otimes \tilde{\partial}_{\kappa^\lambda}\,, \quad \lambda, \eta, \kappa, \theta = (1, .., n), \quad \alpha \in \mathbb{R}
$$

Explicitly, each $\tilde{\Gamma} \in Sec((M \times \mathbb{R})/M)$ gives an injection $(\tilde{\Gamma} \circ \pi_F, I_{FM})$, of FM into $(M \times \mathbb{R}) \times FM$, which is a section of π_1 and Γ coincides with the restriction of Λ_F to this section.

The connection Λ is universal in the following sense. If $\tilde{\Gamma} \in Sec((M \times \mathbb{R})/M)$, then $\tilde{\Gamma}$ is a constant real function on M, so the induced connection $\Gamma = \xi \circ (\tilde{\Gamma} \circ \pi_F, I_{FM}) : FM \longrightarrow JFM$ coincides with restriction on Λ, on the embedding by $(\tilde{\Gamma} \circ \pi_F, I_F M)$ of FM in $(M \times \mathbb{R}) \times_M FM$. So Γ is a pullback of Λ. The universal curvature on the system of α-connections is

$$
\Omega : (M \times \mathbb{R}) \times_M FM \longrightarrow \wedge^2(T^*(M \times \mathbb{R})) \otimes_{FM} V(FM).
$$

So Ω has the explicit form:

$$\Omega = \frac{1}{2}\left(b^k_\kappa\,\alpha\,\tilde{\partial}_{\kappa^\beta} b^m_\omega\,\alpha\,dx^\lambda \wedge dx^\eta + 2\,\partial_a b^m_\omega\,\alpha\,dx^h \wedge dx^\eta\right) \otimes \tilde{\partial}_{\omega^\alpha}\,,$$
$$\text{for } (\kappa, k, \beta, m, \omega, \lambda, \eta, h = 1, .., n), \quad \alpha \in \mathbb{R}.$$

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