Second order structures for sprays and connections on Fréchet manifolds

M. Aghasi[1], A.R. Bahari[1], C.T.J. Dodson[2*],
G.N. Galanis[3] and A. Suri[1]
[1]Department of Mathematics, Isfahan University of Technology,
Isfahan, Iran
[2]School of Mathematics, University of Manchester,
Manchester M13 9PL, UK
[3]Section of Mathematics, Naval Academy of Greece, Xatzikyriakion,
Piraeus 185 39, Greece

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Abstract

Ambrose, Palais and Singer [6] introduced the concept of second order structures on finite dimensional manifolds. Kumar and Viswanath [23] extended these results to the category of Banach manifolds. In the present paper all of these results are generalized to a large class of Fréchet manifolds. It is proved that the existence of Christoffel and Hessian structures, connections, sprays and dissections are equivalent on those Fréchet manifolds which can be considered as projective limits of Banach manifolds. These concepts provide also an alternative way for the study of ordinary differential equations on non-Banach infinite dimensional manifolds. Concrete examples of the structures are provided using direct and flat connections.

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*Email: ctdodson@manchester.ac.uk

1 introduction

The study of infinite dimensional manifolds has received much interest due to its interaction with bundle structures, fibrations and foliations, jet fields, connections, sprays, Lagrangians and Finsler structures ([1],[14],[7], [8], [10], [18] and [30]). In particular, non-Banach locally convex modelled manifolds have been studied from different points of view (see for example [2], [4], [11], [12], [19] and [27]). Fréchet spaces of sections arise naturally as configurations of a physical field and the moduli space of inequivalent configurations of a physical field is the quotient of the infinite-dimensional configuration space $\mathcal{X}$ by the appropriate symmetry gauge group. Typically, $\mathcal{X}$ is modelled on a Fréchet space of smooth sections of a vector bundle over a closed manifold. For example, see Omori [25, 26].

The second order structures introduced by Ambrose et al. [6] for finite dimensional manifolds were extended by Kumar and Viswanath [23] for Banach modelled manifolds. They proved that Hessian structures, sprays, dissections and (linear) connections are in a one-to-one correspondence. However, there these concepts have to be supported by a Christoffel bundle and vector fields. In this paper, following the lines of [23], we first construct the concepts of Christoffel bundle and fields for a class of projective limit Fréchet manifolds. Then, we identify it with the other structures, i.e. connections, Hessian structures and sprays.

One of the main problems in the study of non-Banach modelled manifolds M is the pathological structure of the general linear group $GL(\mathbb{F})$ of a non-Banach space $\mathbb{F}$. $GL(\mathbb{F})$ serves as the structure group of the tangent bundle TM, similar to finite dimensional and Banach cases, but it is not even a reasonable topological group structure within the Fréchet framework (see [16], [18]).
Moreover, for a Fréchet space $\mathbb{F}$, $L(\mathbb{F})$, the space of linear maps on $\mathbb{F}$, is not in general a Fréchet space. The same problem holds for the space of bilinear maps $L^2(\mathbb{F},\mathbb{F}) = \{B; B : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}, B \text{ is linear}\}$.

If one follows the classical procedure to define the notion of Christoffel bundle or Hessian structures, then $L^2(\mathbb{F},\mathbb{F})$ will appear as the corresponding fibre type. As stated in Section 2, these problems are overcome by replacing $L^2(\mathbb{F},\mathbb{F})$ with an appropriate Fréchet space. Another serious drawback in the study of Fréchet manifolds and bundles is the fact that there is no general solvability theory for differential equations ([27]). This problem also can be overcome if we restrict ourselves to the category of those Fréchet manifolds which can be considered as projective limits of Banach corresponding factors. To eliminate these difficulties, we endow TM with a generalized vector bundle structure. (Note that Galanis in [16] proved a similar result but with a different definition for tangent bundle). In the sequel we construct the Christoffel bundles, connections, Hessian structures, sprays and dissections. It is shown in this way that all the results stated in [6] and [23] hold in the category of projective limit manifolds.

Our approach here gives the opportunity to study the problems related to ordinary differential equations that arise via geometric objects on manifolds. For example, geodesics with respect to connections and sprays, and parallel transport are discussed. Finally, the associated structures for flat and direct connections are introduced.

2 Christoffel bundle

Most of our calculus is based on [5] and [24]. Let $\mathbb{E}$ be a real Banach space, M a Hausdorff paracompact smooth manifold and m a point of M. The tangent bundle of M is defined as follows: $TM = \bigcup_{m\in M} T_m M$, where $T_m M$ is considered as the set of equivalence classes of all triples (U, φ, e), where (U, φ) is a chart of M around m and e is an element of the model space $\mathbb{E}$ in which φU lies. TM is a vector bundle on M with structure group $GL(\mathbb{E})$ ([24]).

We summarise our basic notations about a certain rather wide class of Fréchet manifolds, namely those which can be considered as projective limits of Banach manifolds. Let $\{(M_i, \varphi_{ji})\}_{i,j\in\mathbb{N}}$ be a projective system of Banach manifolds with $M = \varprojlim M_i$ such that for every $i \in \mathbb{N}$, M_i is modelled on the Banach space $\mathbb{E}_i$ and $\{\mathbb{E}_i, \rho_{ji}\}_{i\in\mathbb{N}}$ forms a projective system of Banach spaces. Furthermore suppose that for each $m = (m)_{i\in\mathbb{N}} \in M$ there exists a projective system of local charts $\{(U_i, \varphi_i)\}_{i\in\mathbb{N}}$ such that $m_i \in U_i$ and $U = \varprojlim U_i$ is open in M (see [4]).

It is known that for a Fréchet space $\mathbb{F}$, the general linear group $GL(\mathbb{F})$ cannot be endowed with a smooth Lie group structure. It does not even admit a reasonable topological group structure. The problems concerning the structure group of TM can be overcome by the replacement of $GL(\mathbb{F})$ with the following topological group (and in a generalized sense it is also a smooth Lie group):

$$\mathcal{H}_0(\mathbb{F}) = \{(f_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} GL(\mathbb{E}_i) : \varprojlim f_i \text{ exists}\}.$$

More precisely $\mathcal{H}_0(\mathbb{F})$ is isomorphic to the projective limit of the Banach Lie groups

$$\mathcal{H}_0{}^i(\mathbb{F}) = \{(f_1, f_2, ..., f_i) \in \prod_{k=1}^{i} GL(\mathbb{E}_k) : \rho_{jk} \circ f_j = f_k \circ \rho_{jk}, (k \le j \le i)\}.$$

Under these notations the following basic theorems hold (compare with [16]).

Theorem 2.1. *If $\{M_i\}_{i\in\mathbb{N}}$ is a projective system of manifolds then $\{TM_i\}_{i\in\mathbb{N}}$ is also a projective system with limit (set-theoretically) isomorphic to $TM = \varprojlim TM_i$.*

Theorem 2.2. *$TM = \varprojlim TM_i$ has a Fréchet vector bundle structure on $M = \varprojlim M_i$ with structure group $\mathcal{H}_0(\mathbb{F})$.*

Let $L(\mathbb{E}, \mathbb{E})$ be the space of continuous linear maps from a Banach space $\mathbb{E}$ to $\mathbb{E}$ and let $L^2(\mathbb{E}, \mathbb{E})$ be the space of all continuous bilinear maps from $\mathbb{E} \times \mathbb{E}$ to $\mathbb{E}$. For $m \in M$ and every chart (U, φ) at m, consider the triples of the form (U, φ, B) where $B \in L^2(\mathbb{E}, \mathbb{E})$.

Definition 2.3. Two triples (U, φ, B_1) and (V, ψ, B_2) are called equivalent at m if

$$B_2(DF(u).e_1, DF(u).e_2) = DF(u).B_1(e_1, e_2) + D^2F(u)(e_1, e_2), \tag{1}$$

where $u = \varphi m$, $F = \psi \circ \varphi^{-1}$ and $e_1, e_2 \in \mathbb{E}$.

It can be checked that this is an equivalence relation. Each equivalence class is called a Christoffel element at m and a typical element is denoted by γ. Let (U, φ) be a fixed chart at m. Define the mapping

$$\begin{aligned} C_\varphi : C_m &\longmapsto L^2(\mathbb{E}, \mathbb{E}) \\ \gamma &\longmapsto (\varphi m, B) \end{aligned}$$

where C_m is the set of all Christoffel elements at m and $(U, \varphi, B) \in \gamma$. Then C_φ is a bijection, which endows $CM = \bigsqcup_{m\in M} C_m$ with a C^∞-atlas. (For more details see [23]).

From [23] we have the result:

Theorem 2.4. *The family* $\{(CU, C\varphi)$*:* (U, φ) is a chart on $M\}$ is a C^∞-atlas for CM.

We emphasise again at this point that for a Fréchet space $\mathbb{F}$, $L^2(\mathbb{F}, \mathbb{F})$ does not need to be a Fréchet space in general. Hence, the classical procedure for CM for a non-Banach Fréchet manifold M, does not yield a Fréchet manifold (nor bundle) structure. To overcome this obstacle we use the Fréchet space:

$$\mathcal{H}^2(\mathbb{F}, \mathbb{F}) := \{(B_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} L^2(\mathbb{E}_i, \mathbb{E}_i) : \ \varprojlim B_i \text{ exists}\}.$$

$\mathcal{H}^2(\mathbb{F}, \mathbb{F})$ is isomorphic to the projective limit of Banach spaces

$$\mathcal{H}_i^2(\mathbb{F}, \mathbb{F}) := \{(B_1, ..., B_i) \in \prod_{k=1}^{i} L^2(\mathbb{E}_k, \mathbb{E}_k) : \ B_k \circ (\rho_{jk} \times \rho_{jk}) = \rho_{jk} \circ B_j, (k \leq j \leq i)\}.$$

Let $\{M_i\}_{i\in\mathbb{N}}$ be a projective system of Banach manifolds as introduced earlier, B, $\bar{B} \in \mathcal{H}^2(\mathbb{F}, \mathbb{F})$ and $(U = \varprojlim U_i, \varphi = \varprojlim \varphi_i)$, $(V = \varprojlim V_i, \psi = \varprojlim \psi_i)$ two corresponding charts.

Definition 2.5. Two triples $[U, \varphi, B]$ and $[V, \psi, \bar{B}]$ are equivalent if, for every $i \in \mathbb{N}$, $[U_i, \varphi_i, B_i]$ and $[V_i, \psi_i, \bar{B}_i]$ are equivalent.

By these means one can show that CM is endowed with a Fréchet manifold structure modelled on $\mathbb{F} \times \mathcal{H}^2(\mathbb{F}, \mathbb{F})$.

Proposition 2.6. *If* $\{M_i\}_{i\in\mathbb{N}}$ *is a projective system of manifolds and* $\varprojlim CM_i$ *exists then* $\varprojlim CM_i = C(\varprojlim M_i)$ *(set-theoretically).*

Proof. If we consider

$$\begin{array}{rcl} Q : C(\varprojlim M_i) & \longrightarrow & \varprojlim(CM_i) \\ [U, \varphi, B] & \longmapsto & ([U_i, \varphi_i, B_i]_i)_{i\in\mathbb{N}} \end{array}$$

then Q is well defined. Q is one to one since $Q([U, \varphi, B]) = Q([\bar{U}, \bar{\varphi}, \bar{B}])$ yields;

$$[U_i, \varphi_i, B_i]_i = [\bar{U}_i, \bar{\varphi}_i, \bar{B}_i]_i \ , \ i \in \mathbb{N}.$$

Consequently $[U, \varphi, B] = [\varprojlim U_i, \varprojlim \varphi_i, \varprojlim B_i] = \varprojlim[U_i, \varphi_i, B_i]_i = \varprojlim[\bar{U}_i, \bar{\varphi}_i, \bar{B}_i]_i$ $= [\varprojlim \bar{U}_i, \varprojlim \bar{\varphi}_i, \varprojlim \bar{B}_i] = [\bar{U}, \bar{\varphi}, \bar{B}]$. Then Q is also surjective since for every $([U_i, \varphi_i, B_i]_i)_{i\in\mathbb{N}}$ in $\varprojlim(CM_i)$, $Q(a) = ([U_i, \varphi_i, B_i]_i)_{i\in\mathbb{N}}$ where $a = [\varprojlim U_i, \varprojlim \varphi_i, \varprojlim B_i]$.

Therefore, Q is a bijection between CM and $\varprojlim(CM_i)$. □

The functions

$$\begin{array}{rcl} \xi_\alpha : \pi^{-1}(U_\alpha) & \longrightarrow & U_\alpha \times L^2(\mathbb{E}, \mathbb{E}) \\ \gamma & \longmapsto & (m, B); \ \alpha \in I \end{array}$$

with $\gamma \in C_m$, $(U_\alpha, \varphi_\alpha, B) \in \gamma$, define a family of trivializations under which (CM, M, π) becomes a fibre bundle (π is the natural projection).

In the next theorem the concept of (CM, M, π) is generalized to a Fréchet manifold $M = \varprojlim M_i$.

Theorem 2.7. *If $CM = \varprojlim CM_i$ exists, then it admits a Fréchet fibre bundle structure on $M = \varprojlim M_i$ with fibre type $\mathcal{H}^2(\mathbb{F}, \mathbb{F})$.*

Proof. Let $\mathcal{A} = \{(U_\alpha = \varprojlim {U_\alpha}^i, \varphi_\alpha = \varprojlim {\varphi_\alpha}^i)\}$ be an atlas for $M = \varprojlim M_i$. Then, for every $i \in \mathbb{N}$, (CM_i, M_i, π_i) is a fibre bundle with fibres of type $L^2(\mathbb{E}_i, \mathbb{E}_i)$ and trivializations the mappings:

$$\begin{aligned} {\xi_\alpha}^i : \pi_i^{-1}({U_\alpha}^i) &\longrightarrow {U_\alpha}^i \times L^2(\mathbb{E}_i, \mathbb{E}_i) \\ \gamma_i &\longmapsto (m_i, B_i) \end{aligned}$$

Suppose that $\{c_{ji}\}_{i,j\in\mathbb{N}}$, $\{\varphi_{ji}\}_{i,j\in\mathbb{N}}$ and $\{\rho_{ji}\}_{i,j\in\mathbb{N}}$ are the connecting morphisms of the projective systems $CM = \varprojlim CM_i$, $M = \varprojlim M_i$ and $\mathbb{F} = \varprojlim \mathbb{E}_i$ respectively. Since $\varphi_{ji}\pi_j = \pi_i c_{ji}$, $\{\pi_i\}_{i\in\mathbb{N}}$ is a projective system of maps. For every $\alpha \in I$, $\{{\xi_\alpha}^i\}_{i\in\mathbb{N}}$ is a projective system and $\pi = \varprojlim \pi_i : CM \longrightarrow M$ serves as the projection map. On the other hand, $\xi_\alpha := \varprojlim {\xi_\alpha}^i : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathcal{H}^2(\mathbb{F}, \mathbb{F})$ is a diffeomorphism since it is a projective limit of diffeomorphisms. □

For an open subset U in $\mathbb{E}$, define a Christoffel map Γ on U to be a smooth mapping $\Gamma : U \longrightarrow L^2(\mathbb{E}, \mathbb{E})$ and for every chart (U, φ) of M a Christoffel map is locally a smooth mapping $\Gamma_\varphi : \varphi U \longrightarrow L^2(\mathbb{E}, \mathbb{E})$.

Definition 2.8. M is endowed with a Christoffel structure $\{\Gamma_\varphi\}$ if for every choice of charts (U, φ) and (V, ψ) with $U \cap V \neq \emptyset$, the following relation holds true:

$$\Gamma_\psi(v)(DF(u).e_1, DF(u).e_2) = DF(u).\Gamma_\varphi(u)(e_1, e_2) + D^2F(u).(e_1, e_2)$$

where $e_1, e_2 \in \mathbb{E}$, $\varphi m = u$, $\psi m = v$ and $F = \psi \circ \varphi^{-1}$.

For $v, w \in T_m M$ we can express this condition as follows:

$$\Gamma_\psi(v)(v_\psi, w_\psi) = DF(u).\Gamma_\varphi(u)(v_\varphi, w_\varphi) + D^2F(u).(v_\varphi, w_\varphi)$$

where $\varphi m = u$, $v_\psi = DF(u).v_\varphi$, $w_\psi = DF(u).w_\varphi$, $v = [U, \varphi, v_\varphi]$ and $w = [U, \varphi, w_\varphi]$ (see also [23]).

In a similar manner one can define the Christoffel map for the non-Banach case as follows: Let $U = \varprojlim U_i$ be an open subset of $\mathbb{F} = \varprojlim \mathbb{E}_i$. A Christoffel map on $U = \varprojlim U_i$, is a projective limit smooth mapping $\Gamma = \varprojlim \Gamma_i : U \longrightarrow \mathcal{H}^2(\mathbb{F}, \mathbb{F})$. Note that for each chart $(U = \varprojlim U_i, \varphi = \varprojlim \varphi_i)$ of M, $\varprojlim \Gamma_{\varphi_i} := \Gamma_\varphi : \varphi U \longrightarrow \mathcal{H}^2(\mathbb{F}, \mathbb{F})$ defines a Christoffel map on U. Now we can state the following definition for Fréchet manifolds.

Definition 2.9. $M = \varprojlim M_i$ is endowed with a Christoffel structure $\{\Gamma_\varphi = \varprojlim \Gamma_{\varphi_i}\}$, if for every pair of charts $(U = \varprojlim U_i, \varphi = \varprojlim \varphi_i)$ and $(V = \varprojlim V_i, \psi = \varprojlim \psi_i)$ around $m = (m_i)_{i\in\mathbb{N}}$ the following relation is satisfied:

$$\Gamma_\psi(v)(DF(u).e_1, DF(u).e_2) = DF(u).\Gamma_\varphi(u)(e_1, e_2) + D^2F(u).(e_1, e_2),$$

where $e_1 = (e_1^i)_{i\in\mathbb{N}}, e_2 = (e_2^i)_{i\in\mathbb{N}} \in \mathbb{F}$, $\varprojlim \varphi_i m_i = \varprojlim u_i = u$, $\varprojlim \psi_i m_i = \varprojlim v_i = v$ and $F = \varprojlim F_i = \varprojlim \psi_i \circ {\varphi_i}^{-1}$. For $v, w \in T_m M = \varprojlim T_{m_i} M_i$ this condition takes the form

$$\Gamma_\psi(v)(v_\psi, w_\psi) = DF(u).\Gamma_\varphi(u)(v_\varphi, w_\varphi) + D^2F(u).(v_\varphi, w_\varphi)$$

where $\varprojlim \varphi_i m_i = \varprojlim u_i = u$, $v_\psi = \varprojlim DF_i(u_i).v_{\varphi_i}$, $w_\psi = \varprojlim DF_i(u_i).w_{\varphi_i}$, $v = ([U_i, \varphi_i, v_{\varphi_i}]_i)_{i\in\mathbb{N}}$ and $w = ([U_i, \varphi_i, w_{\varphi_i}]_i)_{i\in\mathbb{N}}$.

3 Connections and Hessian structures

A connection on M by Koszul's definition (see [15]) is a smooth mapping

$$\begin{array}{rcl}\nabla : \chi(M) \times \chi(M) & \longrightarrow & \chi(M) \\ (X, Y) & \longmapsto & \nabla_X Y\end{array}$$

such that on every local chart (U, φ) on M, there exists a smooth map $\Gamma_\varphi : \varphi U \longrightarrow L^2(\mathbb{E}, \mathbb{E})$ with

$$(\nabla_X Y)(\varphi m) = DY_\varphi(\varphi m).X_\varphi(\varphi m) - \Gamma_\varphi(\varphi m)(X_\varphi(\varphi m), Y_\varphi(\varphi m));\ \forall m \in U.$$

We prove in the sequel that if ∇ is a connection on M, then $\{\Gamma_\varphi\}$ forms a Christoffel structure on M. Conversely if $\{\Gamma_\varphi\}$ is a Christoffel structure on M and $X, Y \in \chi(U)$, then a connection ∇ can be defined by

$$(\nabla_X Y)(m) = T\varphi^{-1}[DY_\varphi(\varphi m).X_\varphi(\varphi m) - \Gamma_\varphi(\varphi m)(X_\varphi(\varphi m), Y_\varphi(\varphi m))]$$

(see [23]).

Before proceeding to results, it is necessary to prove the following.

Theorem 3.1. *The limit* $\nabla = \varprojlim \nabla_i$ *of a projective system of connections* $\{\nabla_i\}_{i\in\mathbb{N}}$ *is a connection on* $M = \varprojlim M_i$.

Proof. For $i \leq j$, let (U_j, φ_j) be a chart of M_j around m_j and (U_i, φ_i) be a chart of M_i at $\varphi_{ji} m_j = m_i$. Moreover for every $i \in \mathbb{N}$, let $X_{\varphi_i} : \varphi_i U_i \longrightarrow \mathbb{E}_i$ be the local principal part of $X_i \in \chi(M_i)$. Since ∇ is a smooth mapping as a projective limit of smooth factors, to prove the theorem it suffices to check that $\rho_{ji} \circ \nabla_{X_{\varphi_j}} Y_{\varphi_j} = \nabla_{X_{\varphi_i}} Y_{\varphi_i} \circ \rho_{ji}$.
The last equality holds since for $m_j \in U_j$;

$$\begin{aligned} & \rho_{ji} \circ \nabla_{X_{\varphi_j}} Y_{\varphi_j}(\varphi_j m_j) \\ = \ & \underbrace{\rho_{ji} DY_{\varphi_j}(\varphi_j m_j).X_{\varphi_j}(\varphi_j m_j)}_{*} - \underbrace{\rho_{ji}\Gamma_{\varphi_j}(\varphi_j m_j)(X_{\varphi_j}(\varphi_j m_j), Y_{\varphi_j}(\varphi_j m_j))}_{**} \\ = \ & \nabla_{X_{\varphi_i}} Y_{\varphi_i}(\varphi_i m_i) = \nabla_{X_{\varphi_i}} Y_{\varphi_i} \rho_{ji}(\varphi_j m_j) \end{aligned}$$

Note that

$$\begin{aligned} * &= \frac{d}{dt}\rho_{ji} Y_{\varphi_j}(\varphi_j m_j + tX_{\varphi_j}(\varphi_j m_j))_{|t=0} \\ &= \frac{d}{dt} Y_{\varphi_i}\rho_{ji}(\varphi_j m_j + tX_{\varphi_j}(\varphi_j m_j)))_{|t=0} \\ &= \frac{d}{dt} Y_{\varphi_i}(\varphi_i m_i + tX_{\varphi_i}(\varphi_i m_i))_{|t=0} \\ &= DY_{\varphi_i}(\varphi_i m_i).X_{\varphi_i}(\varphi_i m_i) \end{aligned}$$

and

$$\begin{aligned} ** &= \Gamma_{\varphi_i}(\varphi_i m_i)(\rho_{ji} \times \rho_{ji})(X_{\varphi_j}(\varphi_j m_j), Y_{\varphi_j}(\varphi_j m_j)) \\ &= \Gamma_{\varphi_i}(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i), Y_{\varphi_i}(\varphi_i m_i)) \end{aligned}$$

□

Based on Theorem 3.1 we may now establish several important properties.

Theorem 3.2. *If* $\nabla = \varprojlim \nabla_i$ *is a connection on* $M = \varprojlim M_i$*, then* $\{\Gamma_\varphi = \varprojlim \Gamma_{\varphi_i}\}$ *forms a Christoffel structure on* M*.*

Proof. Let $(U{=}\varprojlim U_i,\ \varphi{=}\varprojlim \varphi_i)$, $(V{=}\varprojlim v_i,\ \psi{=}\varprojlim \psi_i)$ be two charts through $m = (m_i)_{i\in\mathbb{N}} \in M$ and $\varprojlim \varphi_i m_i = \varprojlim u_i = u$, $F = \varprojlim F_i = \varprojlim(\psi_i \circ {\varphi_i}^{-1})$. Furthermore, suppose that $X_\varphi = \varprojlim X_{\varphi_i}$, then

$$\begin{aligned}
&[DY_\psi.X_\psi](F(u)) = [D\varprojlim Y_{\psi_i}.\varprojlim X_{\psi_i}](\varprojlim F_i(\varprojlim u_i)) = \varprojlim[[DY_{\psi_i}.X_{\psi_i}](F_i(u_i))] \\
= \;& \varprojlim[DY_{\psi_i}(F_i(u_i)).X_{\psi_i}(F_i(u_i))] = \varprojlim[DY_{\psi_i}(F_i(u_i)).DF_i(u_i).X_{\varphi_i}(u_i)] \\
= \;& \varprojlim[D(Y_{\psi_i}\circ F_i)(u_i).X_{\varphi_i}(u_i)] = \varprojlim[D(DF_i.Y_{\varphi_i}(u_i).X_{\varphi_i}(u_i)] \\
= \;& \varprojlim[D^2F_i(u_i)(X_{\varphi_i},Y_{\varphi_i}) + DF_i(u_i).DY_{\varphi_i}(u_i).X_{\varphi_i}(u_i)].
\end{aligned}$$

But

$$\begin{aligned}
(\nabla_X Y)_\varphi \circ F &= \varprojlim[(\nabla_{X_i}Y_i)_{\varphi_i}\circ F_i] = \varprojlim[(DY_{\psi_i}.X_{\psi_i})\circ F_i - \Gamma_{\psi_i}(X_{\psi_i},Y_{\psi_i})] \\
&= \varprojlim[D^2F_i(X_{\varphi_i},Y_{\varphi_i}) + DF_i.(DY_{\varphi_i}.X_{\varphi_i}) - \Gamma_{\psi_i}(X_{\psi_i},Y_{\psi_i})],
\end{aligned}$$

hence

$$\begin{aligned}
\Gamma_\psi(X_\psi,Y_\psi) = \varprojlim[\Gamma_{\psi_i}(X_{\psi_i},Y_{\psi_i})] = \varprojlim[D^2F_i(X_{\varphi_i},Y_{\varphi_i}) + \\
DF_i.(DY_{\varphi_i}.X_{\varphi_i}) - DF_i.(\nabla_{X_i}Y_i)_{\varphi_i}] = D^2F(X_\varphi,Y_\varphi) + DF.\Gamma_\varphi(X_\varphi,Y_\varphi).
\end{aligned}$$

i.e. $\{\Gamma_\varphi = \varprojlim \Gamma_{\varphi_i}\}$ forms a Christoffel structure on $M = \varprojlim M_i$. □

Remark 3.3. The converse also of Theorem 3.2 can be obtained by setting

$$\begin{aligned}
(\nabla_X Y)(m) = \varprojlim[T\varphi_i^{-1}[DY_{\varphi_i}(\varphi_i m_i).X_{\varphi_i}(\varphi_i m_i) - \\
\Gamma_{\varphi_i}(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i), Y_{\varphi_i}(\varphi_i m_i))]],
\end{aligned}$$

where $\{\Gamma_\varphi = \varprojlim \Gamma_{\varphi_i}\}$ is a Christoffel structure on M. Moreover for $f = \varprojlim f_i \in C^\infty(M)$ and $X, Y \in \chi(M)$, ∇ satisfies the following conditions:
$(i)\nabla$ is real linear in X and Y,
$(ii)\nabla_{fX}Y = f\nabla_X Y$,
$(iii)\nabla_X(fY) = f\nabla_X Y + (Xf)Y$.

In anticipation of the sequel, a Hessian structure on M is a mapping $H : f \longmapsto Hf$, which associates to every $f \in C^\infty(M)$ a covariant 2-tensor Hf on M such that on a local chart (U,φ) of M and for every $X, Y \in \chi(M)$, there exists a smooth map $\Gamma_\varphi : \varphi U \longrightarrow L^2(\mathbb{E},\mathbb{E})$ with

$$\begin{aligned}
[Hf(X,Y)]_\varphi(\varphi m) = D^2 f_\varphi(\varphi m)(X_\varphi(\varphi m), Y_\varphi(\varphi m)) + \\
Df_\varphi(\varphi m).\Gamma_\varphi(\varphi m)(X_\varphi(\varphi m), Y_\varphi(\varphi m)).
\end{aligned}$$

It turns out that Hf is a Hessian structure on M if and only if M admits the Christoffel structure $\{\Gamma_\varphi\}$. Moreover, there is a one-to-one correspondence between Hessian structures and connections given by $Hf(X,Y) = X(Y(f)) - (\nabla_X Y)f$. (For more details see [23]).

Here we study the above results for projective limit manifolds. However, we should consider just the smooth functions and smooth vector fields such that $\mathcal{F}(M) = \{(f_i)_{i\in\mathbb{N}} : f_i : M_i \longrightarrow \mathbb{R}$ is continuous and $\varprojlim f_i$ exists$\}$ and $\mathcal{G}(M) = \{(X_i)_{i\in\mathbb{N}} : X_i$ is a vector field on M_i and $\varprojlim X_i$ exists$\}$ respectively.

Proposition 3.4. *The limit of a projective system of Hessian structures on* $\{M_i\}_{i\in\mathbb{N}}$ *is a Hessian structure on* $M = \varprojlim M_i$.

Proof. For every $i \in \mathbb{N}$, let $f_i \in C^\infty(M_i)$ and $X_i, Y_i \in \chi(M_i)$. Consider a chart (U_i, φ_i) on M_i. Assume that $\Gamma_{\varphi_i} : \varphi_i U_i \longrightarrow L^2(\mathbb{E}_i, \mathbb{E}_i)$ is a smooth map such that

$$[H_i f_i(X_i, Y_i)]_{\varphi_i}(\varphi_i m_i) = D^2 f_{i_{\varphi_i}}(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i), Y_{\varphi_i}(\varphi_i m_i)) + \\ Df_{i_{\varphi_i}}(\varphi_i m_i).\Gamma_{\varphi_i}(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i), Y_{\varphi_i}(\varphi_i m_i)).$$

Hence we must check that for $j \geq i$, $[H_j f_j(X_j, Y_j)]_{\varphi_j} = [H_i f_i(X_i, Y_i)]_{\varphi_i} \circ \rho_{ji}$. For $m_j \in U_j$;

$$[H_j f_j(X_j, Y_j)]_{\varphi_j}(\varphi_j m_j) = \underbrace{D^2 f_{j_{\varphi_j}}(\varphi_j m_j)(X_{\varphi_j}(\varphi_j m_j), Y_{\varphi_j}(\varphi_j m_j))}_{*} + \\ \underbrace{Df_{j_{\varphi_j}}(\varphi_j m_j).\Gamma_{\varphi_j}(\varphi_j m_j)(X_{\varphi_j}(\varphi_j m_j), Y_{\varphi_j}(\varphi_j m_j))}_{**}. \\ = [H_i f_i(X_i, Y_i)]_{\varphi_i}(\varphi_i m_i) = [H_i f_i(X_i, Y_i)]_{\varphi_i} \rho_{ji}(\varphi_j m_j).$$

Note that;

$$\begin{aligned} & Df_{j_{\varphi_j}}(\varphi_j m_j)(X_{\varphi_j}(\varphi_j m_j)) = D(f_{i_{\varphi_i}} \circ \rho_{ji})(\varphi_j m_j)(X_{\varphi_j}(\varphi_j m_j)) \\ = & \frac{d}{dt}(f_{i_{\varphi_i}} \circ \rho_{ji})(\varphi_j m_j + tX_{\varphi_j}(\varphi_j m_j))|_{t=0} \\ = & \frac{d}{dt} f_{i_{\varphi_i}}(\varphi_i m_i + tX_{\varphi_i}\rho_{ji}(\varphi_j m_j))|_{t=0} \\ = & \frac{d}{dt} f_{i_{\varphi_i}}(\varphi_i m_i + tX_{\varphi_i}(\varphi_i m_i))|_{t=0} = Df_{i_{\varphi_i}}(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i)), \end{aligned}$$

and consequently

$$\begin{aligned} * &= D^2(f_j \circ \varphi_j{}^{-1})(\varphi_j m_j)(X_{\varphi_j}(\varphi_j m_j), Y_{\varphi_j}(\varphi_j m_j)) \\ &= D\big(D(f_{i_{\varphi_i}} \circ \rho_{ji})(\varphi_j m_j)(X_{\varphi_j}(\varphi_j m_j))\big)[Y_{\varphi_j}(\varphi_j m_j)] \\ &= D\big(Df_{i_{\varphi_i}}(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i)\big([Y_{\varphi_i}(\varphi_i m_i)] \\ &= D^2 f_{i_{\varphi_i}}(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i), Y_{\varphi_i}(\varphi_i m_i)). \end{aligned}$$

Moreover

$$\begin{aligned} ** &= D(f_{i_{\varphi_i}} \circ \rho_{ji})(\varphi_j m_j).\Gamma_{\varphi_j}(\varphi_j m_j)(X_{\varphi_j}(\varphi_j m_j), Y_{\varphi_j}(\varphi_j m_j)) \\ &= \frac{d}{dt}(f_{i_{\varphi_i}} \circ \rho_{ji})(\varphi_j m_j + t\Gamma_{\varphi_j}(\varphi_j m_j)(X_{\varphi_j}(\varphi_j m_j), Y_{\varphi_j}(\varphi_j m_j))|_{t=0} \\ &= \frac{d}{dt} f_{i_{\varphi_i}}(\varphi_i m_i + t\Gamma_{\varphi_i}(\varphi_i m_i)(\rho_{ji} \times \rho_{ji})(X_{\varphi_j}(\varphi_j m_j), Y_{\varphi_j}(\varphi_j m_j))|_{t=0} \\ &= Df_{i_{\varphi_i}}(\varphi_i m_i).\Gamma_{\varphi_i}(\varphi_i(m_i)(X_{\varphi_i}(\varphi_i(m_i), Y_{\varphi_i}(\varphi_i m_i)). \end{aligned}$$

Hence $\varprojlim [H_i f_i(X_i, Y_i)]_{\varphi_i} = [Hf(X, Y)]_{\varphi}$ where $f \in \mathcal{F}(M)$, $\varphi = \varprojlim \varphi_i$ and $X, Y \in \mathcal{G}(M)$. □

Next, Theorem 3.5 proves that there is a one-to-one correspondence between Hessian structures and connections on Fréchet manifolds.

Theorem 3.5. *Let $\nabla = \varprojlim \nabla_i$ be a connection on $M = \varprojlim M_i$, and $Hf(X,Y) := X(Y(f)) - (\nabla_X Y)f$. Then H is a Hessian structure on M. Conversely the connection which obtained as projective limit of connections arises from a Hessian structure.*

Proof. Let $v, w \in T_m M = \varprojlim T_{m_i} M_i$ and ($U = \varprojlim U_i, \varphi = \varprojlim \varphi_i$ be a chart around $m = (m_i)_{i \in \mathbb{N}}$. Consider vector fields $\varprojlim X_i, \varprojlim Y_i \in \chi(\varprojlim U_i)$ with $\varprojlim X_i(m_i) = v$ and $\varprojlim Y_i(m_i) = w$. Suppose $\nabla = \varprojlim \nabla_i$ be a connection on $M = \varprojlim M_i$, then $\{\Gamma_\varphi = \varprojlim \Gamma_{\varphi_i}\}$ is a Christoffel structure on M. Hence

$$\begin{aligned}
&X(Y(f))(m) - (\nabla_X Y).f(m) = X_\varphi(Y_\varphi(f_\varphi))(\varphi m) - (\nabla_X Y)_\varphi(f_\varphi)(\varphi m)\\
&= \varprojlim [X_{\varphi_i}(Y_{\varphi_i}(f_{\varphi_i}))(\varphi_i m_i) - (\nabla_{X_i} Y_i)_{\varphi_i}(f_{i_{\varphi_i}})(\varphi_i m_i)]\\
&= \varprojlim [D(Df_{i_{\varphi_i}}.Y_{\varphi_i})(\varphi_i m_i).X_{\varphi_i}(\varphi_i m_i) - Df_{i_{\varphi_i}}(\varphi_i m_i).(\nabla_{X_i} Y_i)_{\varphi_i}(\varphi_i m_i)]\\
&= \varprojlim [D^2 f_{i_{\varphi_i}}(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i), (Y_{\varphi_i}(\varphi_i m_i)\\
&+ Df_{i_{\varphi_i}}(\varphi_i m_i).DY_{\varphi_i}(\varphi_i m_i).X_{\varphi_i}(\varphi_i m_i) - Df_{i_{\varphi_i}}(\varphi_i m_i)[DY_{\varphi_i}(\varphi_i m_i).X_{\varphi_i}(\varphi_i m_i)\\
&- \Gamma_{\varphi_i}(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i), (Y_{\varphi_i}(\varphi_i m_i)]]\\
&= \varprojlim [D^2 f_{i_{\varphi_i}}(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i), Y_{\varphi_i}(\varphi_i m_i)) +\\
&Df_{i_{\varphi_i}}(\varphi_i m_i).\Gamma_{\varphi_i}(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i), Y_{\varphi_i}(\varphi_i m_i))]\\
&= \varprojlim [[H_i f_i(X_i, Y_i)]_{\varphi_i}(\varphi_i m_i)] = [Hf(X,Y)]_\varphi(\varphi m).
\end{aligned}$$

Conversely if $Hf = \varprojlim H_i f_i$ is a Hessian structure on $M = \varprojlim M_i$ then $\{\Gamma_\varphi = \varprojlim \Gamma_{\varphi_i}\}$ forms a Christoffel structure on M. Now we have

$$\begin{aligned}
&X(Y(f))(m) - Hf(X,Y)(m) = X_\varphi(Y_\varphi(f_\varphi))(\varphi m) - [Hf(X,Y)]_\varphi(\varphi m)\\
&= \varprojlim [X_{\varphi_i}(Y_{\varphi_i}(f_{\varphi_i}))(\varphi_i m_i) - [H_i f_i(X_i, Y_i)]_{\varphi_i}(\varphi_i m_i)]\\
&= \varprojlim [D^2 f_{i_{\varphi_i}}(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i), (Y_{\varphi_i}(\varphi_i m_i) + Df_{i_{\varphi_i}}(\varphi_i m_i).DY_{\varphi_i}(\varphi_i m_i).X_{\varphi_i}(\varphi_i m_i)\\
&- D^2 f_{i_{\varphi_i}}(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i), Y_{\varphi_i}(\varphi_i m_i))\\
&- Df_{i_{\varphi_i}}(\varphi_i m_i).\Gamma_{\varphi_i}(\varphi_i m_i)(X_{\varphi_i}(\varphi_i m_i), Y_{\varphi_i}(\varphi_i m_i))]\\
&= \varprojlim [Df_{i_{\varphi_i}}(\varphi_i m_i).(\nabla_{X_i} Y_i)_{\varphi_i}(\varphi_i m_i)] = \varprojlim [(\nabla_{X_i} Y_i)_{\varphi_i}(f_{i_{\varphi_i}})(\varphi_i m_i)]\\
&= (\nabla_X Y)_\varphi(f_\varphi)(\varphi m)
\end{aligned}$$

□

4 Sprays

Definition 4.1. A spray ζ is a second order vector field on M such that on a local chart (U, φ) it is determined by a smooth mapping $\Gamma_\varphi : \varphi U \longrightarrow L^2_s(\mathbb{E}, \mathbb{E})$ in the following way:

$$[\zeta(v)]_\varphi(\varphi m, v_\varphi) = (v_\varphi, \Gamma_\varphi(\varphi m)(v_\varphi, v_\varphi));\ m \in U,\ v \in T_m M$$

(see [23]). Note that this definition coincides with the one given in [24].

Theorem 4.2. *The limit of a projective system of sprays on M_i is a spray on $M = \varprojlim M_i$.*

Proof. For every $i \in \mathbb{N}$, let ζ_i be a second order vector field on M_i. Moreover suppose that $(\varprojlim U_i, \varprojlim \varphi_i)$ is a chart of $M = \varprojlim M_i$. Then on the chart (U_i, φ_i) on M_i, ζ_i is determined by the map $\Gamma_{\varphi_i} : \varphi_i U_i \longrightarrow L^2(\mathbb{E}_i, \mathbb{E}_i)$ with the property

$$[\zeta_i(v_i)]_{\varphi_i}(\varphi_i m_i, v_{\varphi_i}) = (v_{\varphi_i}, \Gamma_{\varphi_i}(\varphi_i m_i)(v_{\varphi_i}, v_{\varphi_i}));\ m_i \in U_i,\ v_i \in T_{m_i} M_i.$$

To prove the result, it suffices to check that for $j \geq i$,

$$(\rho_{ji} \times \rho_{ji})[\zeta_j(v_j)]_{\varphi_j} = [\zeta_i(v_i)]_{\varphi_i}(\rho_{ji} \times \rho_{ji}).$$

Indeed for every $m_j \in U_j$ and $v_j = [U_j, \varphi_j, v_{\varphi_j}] \in T_{m_j}M_j$ one obtains;

$$\begin{aligned}(\rho_{ji} \times \rho_{ji})[\zeta_j(v_j)]_{\varphi_j}(\varphi_j m_j, v_{\varphi_j}) &= (\rho_{ji} \times \rho_{ji})(v_{\varphi_j}, \Gamma_{\varphi_j}(\varphi_j m_j)(v_{\varphi_j}, v_{\varphi_j}))\\ &= (v_{\varphi_i}, \Gamma_{\varphi_i}(\varphi_i m_i)(\rho_{ji} \times \rho_{ji})(v_{\varphi_j}, v_{\varphi_j})) = (v_{\varphi_i}, \Gamma_{\varphi_i}(\varphi_i m_i)(v_{\varphi_i}, v_{\varphi_i}))\\ &= [\zeta_i(v_i)]_{\varphi_i}(\varphi_i m_i, v_{\varphi_i}) = [\zeta_i(v_i)]_{\varphi_i}(\rho_{ji}(\varphi_j m_j), \rho_{ji}(v_{\varphi_j}))\\ &= [\zeta_i(v_i)]_{\varphi_i}(\rho_{ji} \times \rho_{ji})(\varphi_j m_j, v_{\varphi_j}).\end{aligned}$$

As mentioned in [23] if ζ_i is a spray on M_i, for every pair of charts (U_i, φ_i) and (V_i, ψ_i) of M_i at m_i, the transformation formula for Γ_{φ_i} is

$$\Gamma_{\psi_i}(\psi_i m_i)(v_{\psi_i}, v_{\psi_i}) = D^2 F_i(\varphi_i)(v_{\varphi_i}, v_{\varphi_i}) + DF_i(\varphi_i m_i).\Gamma_{\varphi_i}(\varphi_i m_i)(v_{\varphi_i}, v_{\varphi_i})$$

where $F_i = \psi \circ \varphi^{-1}$ and $v_i = [U_i, \varphi_i, v_{\varphi_i}] \in T_{m_i}M_i$. Suppose that $\zeta = \varprojlim \zeta_i$ be a spray on $M = \varprojlim M_i$. Then for charts $(U{=}\varprojlim U_i,\ \varphi{=}\varprojlim \varphi_i)$ and $(V{=}\varprojlim V_i,\ \psi{=}\varprojlim \psi_i)$ at $m = (m)_{i\in\mathbb{N}} \in M$ and $v = [U, \varphi, v_\varphi] \in T_m M$:

$$\begin{aligned}\Gamma_\psi(\psi m)(v_\psi, v_\psi) &= \varprojlim \Gamma_{\psi_i}(\psi_i m_i)(v_{\psi_i}, v_{\psi_i})\\ &= \varprojlim [D^2 F_i(\varphi_i m_i)(v_{\varphi_i}, v_{\varphi_i}) + DF_i(\varphi_i m_i).\Gamma_{\varphi_i}(\varphi_i m_i)(v_{\varphi_i}, v_{\varphi_i})]\\ &= D^2 F(\varphi m)(v_\varphi, v_\varphi) + DF(\varphi m).\Gamma_\varphi(\varphi m)(v_\varphi, v_\varphi)\end{aligned}$$

It means that the spray $\zeta = \varprojlim \zeta_i$ defines the Christffel structure $\{\Gamma_\varphi = \varprojlim \Gamma_{\varphi_i}\}$ on $M = \varprojlim M_i$. □

5 Dissections

The concept of dissection is considered next. Kumar and Viswanath [23] established a one-to-one correspondence between dissections of M and Christoffel structures on M for a Banach manifold M. We extend this correspondence to projective limit manifolds.

For $m \in M$, let $G_m := \{f \in C^\infty(U_m) : U_m$ is a neighbourhood of $m\}$ and $G_m^0 := \{f \in G_m : f(m) = 0\}$. Define the space of 1-jets at m, denoted by $J_m M$, to be the set of all equivalence classes in G_m^0, where two functions $f, g \in G_m^0$ are equivalent if on every chart (U, φ) of M, the following relation holds true: $Df_\varphi(\varphi m) = Dg_\varphi(\varphi m)$. In a similar way for every chart (U, φ) of M, one may define $J_m^2 M := \{[f] \in J_m M : D^2 f_\varphi(\varphi m) = D^2 g_\varphi(\varphi m), \forall g \in [f]\}$. If $s \in J_m^2 M$, then the local representation of s on the chart (U, φ) is $s_\varphi = \alpha_\varphi \oplus B_\varphi \in \mathbb{E}^* \oplus L_s^2(\mathbb{E}, \mathbb{R})$ with transformation rule $\alpha_\psi = \alpha_\varphi \circ DG(v)$ and $B_\psi = B_\varphi \circ (DG(v) \times DG(v)) + \alpha_\varphi \circ DG(v) \circ D^2 F(u) \circ (DG(v) \times DG(v))$, where α_φ is the local representation of $\alpha \in T_m^* M$, $G = \varphi \circ \psi^{-1}$, $u = \varphi m$ and $v = \psi m$ (for more details see [23]).

Definition 5.1. A dissection on M is a map that to every $m \in M$ assigns a closed subgroup of $J_m^2 M$ say D_m. This is done in such a way that for every chart (U, φ) there exists a smooth mapping $\Gamma_\varphi : \varphi U \longrightarrow L_s^2(\mathbb{E}, \mathbb{E})$ such that $B_\varphi = \alpha_\varphi \circ \Gamma_\varphi(u)$ for $s \in D_m$ and $s_\varphi = \alpha_\varphi \oplus B_\varphi$. In other words $[D_m]_\varphi = \{\alpha \oplus \alpha \circ \Gamma_\varphi(u) : \alpha \in \mathbb{E}^*\}$ ([23]).

We extend Kumar and Viswanath's results to projective limit Fréchet manifolds.

Proposition 5.2. *If $\{M_i\}_{i\in\mathbb{N}}$ is a projective system of manifolds and $\varprojlim J^2_{m_i}M_i$ exists then $\varprojlim J^2_{m_i}M_i = J^2_{(m_i)}\varprojlim M_i$ (set-theoretically).*

Proof. Let $G_m := \{(f_i)_{i\in\mathbb{N}}; f_i : U_{m_i} \longrightarrow \mathbb{R}$ is continuous and $\varprojlim f_i$ exists$\}$ and $G^0_m := \{(f_i)_{i\in\mathbb{N}} \in G_m : f_i(m_i) = 0,\ \forall i \in \mathbb{N}\}$. By defining

$$\begin{array}{rcl} p : J^2_m M & \longrightarrow & \varprojlim J^2_{m_i}M_i \\ {[f,m]} & \longmapsto & ([f_i, m_i]_i)_{i\in\mathbb{N}} \end{array}$$

It can be checked that p is well defined; moreover, p is one to one since $p[f,m] = p[g,m]$ yields

$$[f_i, m_i]_i = [g_i, m_i]_i,\ \ i \in \mathbb{N}.$$

Hence $[f,m] = [\varprojlim f_i, (m_i)_{i\in\mathbb{N}}]_i = \varprojlim[f_i, m_i]_i = \varprojlim[g_i, m_i]_i = [\varprojlim g_i, (m_i)_{i\in\mathbb{N}}] = [g,m]$.
Furthermore p is surjective. In fact if $([f_i, m_i]_i)_{i\in\mathbb{N}}$ is an arbitrary element of $\varprojlim J^2_{m_i}M_i$, we define $a = [\varprojlim f_i, (m_i)_{i\in\mathbb{N}}]$. Then $p(a) = ([f_i, m_i]_i)_{i\in\mathbb{N}}$ and therefore p is an isomorphism between $J^2_m M$ and $\varprojlim J^2_{m_i}M_i$. □

Theorem 5.3. *The limit of a projective system of dissections of $\{M_i\}_{i\in\mathbb{N}}$ is a dissection of $\varprojlim M_i = M_i$.*

Proof. For every $i \in \mathbb{N}$, suppose D_{m_i} is the closed subgroup of $J^2_{m_i}M_i$ with the above mentioned properties. Moreover for $j \geq i$,

$$B_{\varphi_j} = \alpha_{\varphi_j} \circ \Gamma_{\varphi_j}(u_j) = (\alpha_{\varphi_i} \circ \rho_{ji}) \circ \Gamma_{\varphi_j}(u_j) = \alpha_{\varphi_i} \circ (\Gamma_{\varphi_i}(u_i) \circ (\rho_{ji} \times \rho_{ji}))$$
$$= B_{\varphi_i} \circ (\rho_{ji} \times \rho_{ji}).$$

Therefore $\varprojlim D_{m_i}$ exists and it is a dissection on $M = \varprojlim M_i$. □

If $\varprojlim D_{m_i}$ is a dissection of $\varprojlim M_i = M$ and ($U{=}\varprojlim U_i$, $\varphi{=}\varprojlim \varphi_i$), ($V{=}\varprojlim V_i$, $\psi{=}\varprojlim \psi_i$) are two charts at $m = (m_i)_{i\in\mathbb{N}} \in M$, then

$$\begin{aligned} \Gamma_\psi(v) = \varprojlim \Gamma_{\psi_i}(v_i) &= \varprojlim[D^2F_i(u_i) \circ (DG_i(v_i) \times DG_i(v_i)) \\ &\quad + DF_i(u_i) \circ \Gamma_{\varphi_i}(u_i) \circ (DG_i(v_i) \times DG_i(v_i))] \\ = D^2F(u) \circ (DG(v) \times DG(v)) &+ DF(u) \circ \Gamma_\varphi(u) \circ (DG(v) \times DG(v)), \end{aligned}$$

which precisely coincides with the Christoffel structures $\{\Gamma_\varphi = \varprojlim \Gamma_{\varphi_i}\}$. (For more details see [23].) Hence we get the following result.

Corollary 5.4. *There is one-to-one correspondence between dissections and Christoffel structures on $M = \varprojlim M_i$.*

6 Examples

Example 6.1. The direct connection
Let G be a Banach Lie group with the model space $\mathbb{E}$. Consider the mapping $\mu : G \times \eth \longrightarrow TG$ given by $\mu(m,v) = T_e\lambda_m(v)$, where λ_m is the left translation on G and $\eth$ is the Lie algebra of G. According to Vassiliou [31], there exists a unique connection ∇^G on G which is $(\mu, id_G)-$related to the canonical flat connection on the trivial bundle $L = (G \times \eth, pr_1, G)$. Locally the Christoffel symbols Γ^G of ∇^G are given by

$$\Gamma^G_\varphi(x)(a,b) = Df_\varphi(x)(a, f_\varphi^{-1}(m)(b));\ x \in \varphi U,\ a,b \in \mathbb{E}$$

where f_φ is the local expression of the isomorphism $T_e\lambda_x : T_eG \longrightarrow T_xG$ and (U,φ) chart of G. If $G = \varprojlim G_i$ is obtained as a projective limit of Banach Lie groups and ∇^{G_i} is the direct connection on $L^i = (G_i \times \eth_i, pr_1, G_i)$, then $\nabla^G = \varprojlim \nabla^{G_i}$ is exactly the direct connection on $L = (\varprojlim G_i \times \varprojlim \eth_i, pr_1, \varprojlim G_i)$ [21]. Also, ∇^G determines a unique spray on $G = \varprojlim G_i$ locally given by

$$[\zeta^G(v)]_\varphi(\varphi m, v_\varphi) = (v_\varphi, \Gamma^G_\varphi(\varphi m)(v_\varphi, v_\varphi));\ m \in U,\ v \in T_mG.$$

Moreover, using ∇^G the Christoffel structure $\{\Gamma_\varphi\}$ and Hessian structure H^G are obtained where H^G is locally given by

$$[H^G f(X,Y)]_\varphi(\varphi m) = D^2 f_\varphi(\varphi m)(X_\varphi(\varphi m), Y_\varphi(\varphi m)) + Df_\varphi(\varphi m).\Gamma^G_\varphi(\varphi m)(X_\varphi(\varphi m), Y_\varphi(\varphi m)).$$

Example 6.2. The flat connection
Let $M = \mathbb{E}$ with the global chart $(\mathbb{E}, id_\mathbb{E})$. The canonical flat connection ∇^C on the trivial bundle $(M \times \mathbb{E}, pr_1, M)$ is locally given by the Christoffel structure $\{\Gamma^C\}$, where $\Gamma^C(x)(u) = 0$, for every $(x,u) \in \mathbb{E} \times \mathbb{E}$. Let $M = \mathbb{F} = \varprojlim \mathbb{E}_i$ and consider it with the global chart $(\mathbb{F}, id_\mathbb{F}) = \varprojlim(\mathbb{E}_i, id_{\mathbb{E}_i})$. For the canonical flat connection $\Gamma^C = \varprojlim \Gamma^C_i$ on $(M \times \mathbb{F}, pr_1, M)$, the spray ζ^C and the Hessian structure H^C are given by

$$[\zeta^C(v)]_\varphi(\varphi m, v_\varphi) = (v_\varphi);\ m \in U,\ v \in T_mM$$

and

$$[H^C f(X,Y)]_\varphi(\varphi m) = D^2 f_\varphi(\varphi m)(X_\varphi(\varphi m), Y_\varphi(\varphi m)).$$

7 Ordinary differential equations

A curve $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ is called autoparallel or a geodesic with respect to the connection ∇ if $\nabla_{T\gamma}T\gamma = 0$ ([32]). Let (U,φ) be a local chart on M and set $\gamma_\varphi := \varphi \circ \gamma : (-\varepsilon,\varepsilon) \longrightarrow \mathbb{E}$, $\gamma'_\varphi(t) := T\gamma_\varphi : (-\varepsilon,\varepsilon) \longrightarrow T\mathbb{E}$.

In this case the local expression of $\nabla_{T\gamma}T\gamma = 0$ takes the form:

$$\nabla_{T\gamma_\varphi}T\gamma_\varphi(\gamma_\varphi(t) = D\gamma'_\varphi(t).\gamma'_\varphi(t) - \Gamma_\varphi(\gamma_\varphi(t))[\gamma'_\varphi(t), \gamma'_\varphi(t)] = 0.$$

Every spray is a second order vector field, hence every integral curve of ζ is the canonical lifting of $\pi \circ \beta$, so $T(\pi \circ \beta) = \beta$. The curve $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$

is a geodesic spray with respect to ζ if $T\gamma$ is an integral curve for ζ, namely, $T_{T_t\gamma(v_t)}T_t\gamma(v_t) = \zeta T_t\gamma(v_t)$, where $v_t \in T_t\mathbb{R}$ with $pr_2(v_t) = 1$. In local charts we have;

$$\big(\zeta(T_t\gamma(v_t))\big)_\varphi(\gamma_\varphi(t), D_t\gamma_\varphi(v_t)) = \big(\gamma_\varphi(t), \Gamma_\varphi(\gamma_\varphi(t))[D_t\gamma_\varphi(v_t), D_t\gamma_\varphi(v_t)]\big).$$

and

$$(T_{T_t\gamma(v_t)}T_t\gamma(v_t))_\varphi = \big(D_t\gamma_\varphi(v_t), D_{D_t\gamma_\varphi(v_t)}D_t\gamma_\varphi(v_t, v_t)\big) := (\gamma'_\varphi(t), \gamma''_\varphi(t))$$

So γ must satisfy the (local) equation

$$\gamma''_\varphi(t) = \Gamma_\varphi(\gamma_\varphi(t))(\gamma'_\varphi(t), \gamma'_\varphi(t)).$$

Consequently the following theorem holds for Banach modelled manifolds.

Theorem 7.1. *Let ζ be the spray assigned to ∇. There is a one-to-one correspondence between geodesics of ∇ and geodesic sprays of ζ.*

Here we try to generalize this to the case of Fréchet manifolds where difficulties arise due to intrinsic problems of the model spaces of these manifolds and mainly due to the inability to solve general differential equations (see [3], [17] and [27]). We show that if one focuses on the category of projective limit manifolds, then similar results can be obtained.

Theorem 7.2. *Let $M = \varprojlim M_i$ and $\zeta = \varprojlim \zeta_i$ be a spray on M with k-Lipschitz local components. Let $x_0 \in M$ and $y_0 \in T_{x_0}M$. If for a chart (U, φ) around x_0, $M_\varphi = sup\{\big(p_i(x_0)^2 + p_i(\Gamma_\varphi(x_0)[y_0, y_0])^2\big)^{1/2};\ i \in \mathbb{N}\} < \infty$, then there exists a locally unique geodesic spray $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ such that $\gamma(0) = x_0$, $T_t\gamma(0) = y_0$ and $\varepsilon > 0$ is independent of the index i.*

Proof. Let $\zeta : TM \longrightarrow TTM$ be a spray. Consider $\{\zeta_i\}_{i\in\mathbb{N}}$, $x_0 = (x_{0i})_{i\in\mathbb{N}} \in \varprojlim M_i$ and $y_0 = (y_{0i})_{i\in\mathbb{N}} \in \varprojlim T_{x_{0i}}M_i$. For every $i \in \mathbb{N}$, ζ_i is a spray on M_i. Since M_i is a Banach manifold, by the existence theorem for ordinary differential equations, there exists $\gamma_i : (-\varepsilon_i, \varepsilon_i) \longrightarrow M_i$ with

$$\gamma_i{}''_{\varphi_i}(t) = \Gamma_{\varphi_i}(\gamma_{\varphi_i}(t))[\gamma_i{}'_{\varphi_i}(t), \gamma_i{}'_{\varphi_i}(t)], \tag{2}$$

satisfying $\gamma_i(0) = x_{0i}$ and $T_{t\gamma_i}(0) = y_{0i}$. For $j \geq i$, we claim that $\varphi_{ji} \circ \gamma_j = \gamma_i$ and consequently $\{\gamma_i\}_{i\in\mathbb{N}}$ forms a projective system of curves on $\{M_i\}_{i\in\mathbb{N}}$ with the limit $\gamma = \varprojlim \gamma_i$. Note that

$$\begin{aligned}(\varphi_i \circ \varphi_{ji} \circ \gamma_{j_{\varphi_j}})''(t) &= (\rho_{ji} \circ \varphi_j \circ \gamma_{j_{\varphi_j}})''(t) = \rho_{ji}(\varphi_j \circ \gamma_{j_{\varphi_j}})''(t)) = \rho_{ji}\Gamma_{\varphi_j}(\gamma_{j_{\varphi_j}}(t)) \\ (\gamma_j{}'_{\varphi_j}(t), \gamma_j{}'_{\varphi_j}(t)) &= \Gamma_{\varphi_i}((\rho_{ji} \circ \varphi_j \circ \gamma_{j_{\varphi_j}})(t))[(\rho_{ji} \circ \varphi_j \circ \gamma_{j_{\varphi_j}})'(t), (\rho_{ji} \circ \varphi_j \circ \gamma_{j_{\varphi_j}})'(t)] \\ &= \Gamma_{\varphi_i}((\varphi_i \circ \varphi_{ji} \circ \gamma_{j_{\varphi_j}})(t))[(\varphi_i \circ \varphi_{ji} \circ \gamma_{j_{\varphi_j}})'(t), (\varphi_i \circ \varphi_{ji} \circ \gamma_{j_{\varphi_j}})'(t)].\end{aligned}$$

Moreover $(\varphi_{ji}\circ\gamma_j)(0) = \varphi_{ji}(x_{0j}) = x_{0i}$ and $T_t(\varphi_{ji}\circ\gamma_j)(0) = y_{0i}$. By uniqueness of solutions for ordinary differential equations on Banach spaces (manifolds) we have $\varphi_{ji} \circ \gamma_j = \gamma_i$ and consequently $\gamma = \varprojlim \gamma_i$ exists. Furthermore

$$T_{T_t\gamma(v_t)}T_t\gamma(v_t) = \{T_{T_t\gamma_i(v_t)}T_t\gamma_i(v_t)\}_{i\in\mathbb{N}} = \{\zeta_i(T_t\gamma_i(v_t))\}_{i\in\mathbb{N}} = \zeta(T_t\gamma(v_t)).$$

According to Theorem 9.1, ε does not converge to 0 and consequently there exists $\varepsilon > 0$ such that $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ is a local geodesic spray with respect to ζ.

Let $\beta : (-\varepsilon_1, \varepsilon_1) \longrightarrow M$ be another curve such that $T_{T_t\beta(v_t)}T_t\beta(v_t) = \zeta(T_t\beta(v_t))$, satisfying in the above boundary conditions. For every $i \in \mathbb{N}$, $\beta_i = \psi_i \circ \beta$ satisfies in equation (2) with $\beta_i(0) = x_{0i}$ and $T_t\beta_i(0) = y_{0i}$. Hence $\beta_i = \gamma_i$ and consequently $\beta = \varprojlim \beta_i = \varprojlim \gamma_i = \gamma$ on the intersection of their domains. □

Finally in a similar way one can prove the theorem for geodesics with respect to the connection ∇. As a conclusion we can state the following corollary.

Corollary 7.3. *For a projective limit manifold $M = \varprojlim M_i$ there is a one-to-one correspondence between (linear) connections and sprays. Moreover, the geodesics of ∇ are geodesic sprays of ζ.*

8 Parallel translation

Vilms [32] defines a connection on M as a vector bundle morphism $\nabla : T(TM) \longrightarrow TM$. So ∇ is fully determined by its local components, called Christoffel symbols, denoted by $\{\Gamma_\alpha\}_{\alpha\in I}$ corresponding to an atlas of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ of M. Then, $\Gamma_\varphi : \varphi U \longrightarrow L^2(\mathbb{E}, \mathbb{E})$, and for two charts (U, φ) and (V, ψ) at $m \in M$, $e_1, e_2 \in \mathbb{E}$, $u = \varphi(m), v = \psi(m)$, $F = \psi \circ \varphi^{-1}$ we have

$$\Gamma_\psi(v)((DF(u).e_1, DF(u).e_2) = DF(u).\Gamma_\varphi(u)(e_1, e_2) + D^2F(e_1, e_2).$$

Clearly, our definition coincides with the above; we next consider parallel transport of vectors along a curve.

Theorem 8.1. *Given $\nabla : T(TM) \longrightarrow TM$ a connection on (TM, M, π), take a smooth curve $c : (a, b) \longrightarrow M$ with $0 \in (a, b), c(0) = x$. Then, there is a neighbourhood U of $T_xM \times \{0\} \subseteq T_xM \times (a, b)$ and a smooth mapping $\bar{c} : U \longrightarrow TM$ such that:*
(i) $\pi(\bar{c}(u_x, t)) = c(t)$ and $\bar{c}(u_x, 0) = u_x$,
(ii) $\nabla(\frac{d}{dt}(\bar{c})(u_x, t)) = 0$.

Proof. For every (U, φ) chart of M, $\nabla(\frac{d}{dt}\bar{c}(u_x, t)) = 0$, locally gives $-\Gamma_\alpha(c(t))\big(\frac{d}{dt}c(t), \gamma(y, t)\big) + \frac{d}{dt}\gamma(y, t) = 0$, where $T\varphi(\bar{c}(c, T\varphi^{-1}(x, y), t)) := (c(t), \gamma(y, t))$ (i.e. $\gamma : \mathbb{E} \times (a, b) \longrightarrow S$). For M a Banach manifold, by the existence theorem for differential equations, $\bar{c}$ always exists. □

Using our method one can prove a similar theorem for parallel transport along curves in the category of projective limit manifolds. The equivalence of linear connections with sprays means that parallel transport is equivalently determined by a spray [22].

Example 8.2. Geodesics on the diffeomorphism group of the circle
The main reference for this example is Constantin and Kolev [9]. Let $D = Diff(\mathbb{S}^1)^+$ be the group of all smooth orientation-preserving diffeomorphisms of the circle $\mathbb{S}^1$. We can endow D with a smooth manifold structure based on the Fréchet space $C^\infty(\mathbb{S}^1)$.

Moreover a right invariant weak Riemannian metric on D is defined. Note that $C^\infty(\mathbb{S}^1) = \bigcap_{n\geq 2k+1} H^n(\mathbb{S}^1)$ where $H^n(\mathbb{S}^1)$, $n \geq 0$ is the space $L^2(\mathbb{S}^1)$ of all square integrable functions f with the distributional derivatives up to order n, ∂_x^i with $i = 0, 1, ..., n$, in $L^2(\mathbb{S}^1)$. $H^n(\mathbb{S}^1)$, $n \geq 0$ is a Hilbert space with the norm

$$\|f\|_n^2 = \sum_{i=0}^{n} \int_S (\partial_x^i f)^2(x) dx.$$

The main difference of this example for our method lies in the existence of a metric and this allows us to prove the length minimizing property of geodesics.

We move this problem to the projective limit framework. In this special case like [28] the connecting morphisms of the model space are inclusions

$$\begin{aligned} \rho_{n+1,n} : H^{n+1} &\hookrightarrow H^n \\ f &\longmapsto f \end{aligned}$$

The meaning of this morphism is clear, namely if $f \in H^{n+1}$ with the norm on H^{n+1} then $\sum_{i=0}^{n+1} \int_S (\partial_x^i f)^2(x) dx < \infty$. Clearly $\sum_{i=0}^{n} \int_S (\partial_x^i f)^2(x) dx < \infty$, so the function f belongs precisely to H^n. Consequently if $f \in \bigcap_{n\geq 2k+1} H^n(\mathbb{S}^1)$ then $(f) \in \varprojlim H^n(\mathbb{S}^1)$ and, conversely, $C^\infty(\mathbb{S}^1) = \bigcap_{n\geq 2k+1} H^n(\mathbb{S}^1) = \varprojlim_{n\geq 2k+1} H^n(\mathbb{S}^1)$.

For $\varphi_0 \in D$ define $U_0 = \{\varphi \in D : \| \varphi - \varphi_0 \|_{C^0(\mathbb{S}^1)} < 1/2\}$ and $u : u_0 \longrightarrow C^\infty(\mathbb{S}^1)$ with $u(x) = \frac{1}{2\pi i} ln\big(\overline{\varphi_0(x)}\varphi(x)\big)$, $x \in S$. Then (U_0, ψ_0) is a local chart with $\psi_0(\varphi) = u$ and change of charts given by $\psi_2 \circ \psi_1^{-1} = u_1 + \frac{1}{2\pi i} ln(\overline{\varphi_2}\varphi_1)$. Note that $\psi_2 \circ \psi_1^{-1} : \psi_1(u_1) \subseteq C^\infty(\mathbb{S}^1) \longrightarrow \psi_2(u_2) \subseteq C^\infty(\mathbb{S}^1)$ can be recognized as the projective limit on Hilbert components, say $(\psi_2 \circ \psi_1^{-1})_i : H^i(\mathbb{S}^1) \longrightarrow H^i(\mathbb{S}^1)$, $(\psi_2 \circ \psi_1^{-1}) = \varprojlim (\psi_2 \circ \psi_1^{-1})_i$. These maps are called k-Lipschitz and so $(\psi_2 \circ \psi_1^{-1})$. This structure endows D with a smooth manifold structure based on the Fréchet space $C^\infty(\mathbb{S}^1)$.

Let $k \geq 0$, for $n \geq 0$ define the linear seminorms $A_k : H^{n+2k}(\mathbb{S}^1) \longrightarrow H^n(\mathbb{S}^1)$ with $A_k = 1 - \frac{d^2}{dx^2} + ... + (-1)^k \frac{d^{2k}}{dx^{2k}}$. This enables us to define the bilinear operator $B_k : C^\infty \times C^\infty \longrightarrow C^\infty$ with $B_k(u, v) = A_k{}^{-1}(2v_x A_k(u) + vA_k(u_x))$ $u, v \in C^\infty$. Note that $B = \varprojlim_{n\geq 2k+1} B_k{}^n$ where $B_k{}^n : H^n(\mathbb{S}^1) \times H^n(\mathbb{S}^1) \longrightarrow H^{n-2k}(\mathbb{S}^1)$. As stated in [9], Theorem 1, there exists a unique linear (Riemannian) connection ∇^k on D.

If $\varphi : J \longrightarrow D$ is a C^2-curve satisfying the autoparallel equation with respect to the linear connection ∇^k, then

$$u_t = B_k(u, u), \; t \in J$$

where $u = \varphi_t \circ \varphi^{-1} \in T_{Id}D \equiv C^\infty(\mathbb{S}^1)$. The term autoparallel rather than geodesic is better since there is no underlying Riemannian metric. However the utilization of a weak Riemannian metric is an issue that remains open. Since $B_k = \varprojlim B_k{}^i$, is the projective system of bilinear maps (as Christoffel symbols) we can endow D with the linear connection $\nabla_k = \varprojlim \nabla_k{}^i$. Given an initial value we obtain the unique autoparallel $\varphi : J \longrightarrow D$ obtained as the projective limit on Hilbert components. The problem is much easier than the general case. Specifically, let the solution on the $H^n(\mathbb{S}^1)$, $n \geq 2k+1$ have the manifold domain $[0, T_n)$ with $T_n > 0$. If $T_n \leq T_{\leq 2k+1}$ then $T_n = T_{2k+1}$ for all $n \geq 2k+1$ i.e. the solution φ_n on $H^n(\mathbb{S}^1)$, $n \geq 2k$ defined on $[0, T_{2k+1})$ for every n.

Note that in the general case there is no way to model the diffeomorphism group of a manifold M on a Banach space. However, there is the possibility to view $Diff(M)$ as a projective limit of Hilbert manifolds ([29]). Moreover, the existence of the geodesics in the general case of $Diff(M)$ is an open question, so using the proposed technique with an appropriate choice of imposed metric may yield some results.

9 Appendix: Existence and uniqueness theorem for second order ordinary differential equations on Fréchet spaces

Start with the assumptions of [20]. Namely, let $\mathbb{F}$ be a Fréchet space and $\{p_i\}_{i\in\mathbb{N}}$ be a countable family of seminorms which determine the topology of $\mathbb{F}$.

Theorem 9.1. *Let $\mathbb{F}$ be a Fréchet space and $\Phi : \mathbb{R}\times\mathbb{F}\times\mathbb{F} \longrightarrow \mathbb{F}$ a projective limit k-Lipschitz mapping. For the second order differential equation*

$$x'' = \Phi(t, x, x') \tag{3}$$

with the initial condition (t_0, x_0, y_0), if there exists a constant $\tau \in \mathbb{R}^+$ such that

$$M = sup\{\left(p_i(y_0)^2 + p_i(\Phi(t, x_0, y_0))^2\right)^{1/2};\ i \in \mathbb{N}, t \in [t_0 - \tau, t_0 + \tau]\} < \infty$$

and $a = min\{\tau, \frac{1}{M+k}\}$, then (2) has a unique solution on $I = [t_0 - a, t_0 + a]$.

Proof. If we set $x' = y,\ x' = y,\ y' = \Phi(t, x, y)$. Denoting $z = (x, y)$ one takes:

$$z' = (x, y)' = (x', y') = (y, \Phi(t, x, y)) = \tilde{\Phi}(t, z) \tag{4}$$

where $\tilde{\Phi} : \mathbb{R}\times\mathbb{F}\times\mathbb{F}\times\mathbb{F} \longrightarrow \mathbb{F}\times\mathbb{F}$, is also a k-Lipschitz mapping. Since

$$\left(p_i(y_0)^2 + p_i(\Phi(t, x_0, y_0))^2\right)^{1/2} = p_i(y_0, \Phi(t, x_0, y_0)) = p_i(\tilde{\Phi}(t, z_0));$$

and

$$M = sup\{p_i(\tilde{\Phi}(t, z_0)) = \left(p_i(y_0)^2 + p_i(\Phi(t, x_0, y_0))^2\right)^{1/2};\ i \in \mathbb{N}, t \in [t{-}0{-}\tau, t{-}0{+}\tau]\} < \infty$$

by Theorem 3 in [20], (4) has a unique solution on $I = [t_0 - a, t_0 + a]$ such that $a = min\{\tau, \frac{1}{M+k}\}$. Hence there exists also a solution for (3) say $z : I \longrightarrow \mathbb{F}\times\mathbb{F}$. If $z = (z_1, z_2)$ then, z_1 and z_2 are unique solution for $x' = y$ and $y' = \Phi(t, x, y)$ respectively on I. Consequently ${z_1}' = y,\ y' = \Phi(t, z_1, y)$ i.e.

$${z_1}'' = \Phi(t, z_1, z_1') \text{ on } I.$$

Note that the interval I is independent of the index i. For each $i \in \mathbb{N}$ from the equation

$$x_i'' = \Phi_i(t, x_i, x_i')$$

with the initial condition $(t_0, x_{0i}, {y_0}_i')$ we have the unique solution x_i. On the other hand for $i \leq j$, $f_{ji}\circ x_j$ is also a solution of (4) with $f_{ji}\circ x_j(t_0) = x_{0i}$ and $(f_{ji}\circ x_j)'(t_0) = y_{0i}$. Hence $f_{ji}\circ x_j = x_i$ for $i \leq j$, i.e. $x = \varprojlim x_i$ can be defined. Moreover

$$x'' = (x_i'')_{i\in\mathbb{N}} = (\Phi_i(t, x_i, x_i'))_{i\in\mathbb{N}} = \Phi(t, x, x'),$$

i.e. $\varprojlim x_i$ is a solution for (2). The uniqueness of x follows from the uniqueness of solution for Banach components. ∎

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