Analytic solution to the Martinez dewatering equations for roll gap formers

T. Boxer* C. T. J Dodson† W. W Sampson‡

March 8, 2000

Abstract

A model of the roll dewatering process on a twin–wire paper machine was recently published by Martinez. The model equations were solved numerically, comparisons were made with machine measurements and the limitations of the model were stated. We give the analytic solution to the Martinez equations and reveal an implicit, and previously unstated, assumption of constant pressure in the model. The analytic solution is an explicit statement of the dependence of flow rate on machine and furnish parameters. An experimental measurement of the pressure trace is not required in order to apply the model. The machine conditions of the trials conducted by Martinez are substituted into the new analytic formula for the flow rate. Good agreement is found between theory and experiment.

Introduction

In the last 5 years there have been many contributions to the modelling of twin–wire forming. Most have been concerned with the blade forming process. Norman [1] determined the dewatering force imparted by the blade. Zhao and Kerekes [2] constructed a linear 1D mathematical model to predict the shape of the pressure pulse over a thin blade. Since then there has been a rapid development in the accuracy and complexity of blade models: Zahrai and Bark [3], Green and Kerekes [4], Nigam and Bark [5], Zahrai *et al.* [6], and Green*et al.* [7]. Numerical solutions have now been generated to the full non–linear equations for 2D flow over an arbitrary shaped blade. A valuable additional contribution was made by Green [8] who published simple analytic expressions to describe blade dewatering. These results are more accessible and easily applied than the numerical solutions, and they provide an understanding of the fundamentals of the process.

*Diagnostic Engineer, Arjo Wiggins Ltd., Butler's Court, Beaconsfield, Bucks., HP9 1RT, UK.
†Professor, Department of Mathematics, UMIST, PO Box 88, Manchester, M60 1QD, UK.
‡Lecturer, Department of Paper Science, UMIST, PO Box 88, Manchester, M60 1QD, UK.

Recently Martinez [9] presented a model of roll dewatering. This work improved on previous studies by including the compressibility of the fibre sheet. The pressure trace was measured on an experimental machine and this was combined with careful modelling of the sheet resistance to determine the dewatering rate. Numerical solutions were generated and compared with measurements of flow rate in a series of 19 trials on an experimental machine; the model predicted the measured behaviour very well.

In this paper the limitations of the Martinez model are explained more precisely and the analytic solution is presented. This provides a simple and powerful formula for the flow rate around a roll. It is shown that there is an implicit, previously unstated, assumption in the Martinez model of constant pressure; hence, measurement of the roll pressure is unnecessary. We provide the analytic solution to the equations by first considering a special case that reduces the non–linearity, and then extending to the general case. The simple explicit formulae for flow rate through each fabric are used to simulate the trial results from [9]. The analytic solution accurately predicts the observed effects of changes in machine conditions.

Model equations

The mathematical model of roll dewatering presented by Martinez [9] is summarised below. The equations for fabric separation are not used in our derivation, which considers only conditions of constant pressure, but are given here for completeness. The derivation and assumptions up to this point are clearly described in [9] and are therefore not reproduced here.

Fabric separation, $G(x)$:

$$\frac{d^2G}{dx^2} = \frac{L^2}{R}\left(1-\left(\frac{Lx}{R}\right)^2\right)^{-\frac{3}{2}} - \sigma_1(x)\frac{L^2}{T} \tag{1}$$

$$G(0) = G_0 \ , \ G(1) = G_0\frac{U_j}{U_w} - \frac{F_1+F_2}{U_wW} \tag{2}$$

where x is the dimensionless position in the machine direction, scaled by the length of the forming zone, L. The forming fabrics of width W are in tension T around the forming roll of radius R. The initial fabric separation, G_0, is taken to be the width of the slice opening. The jet of stock from the slice has speed U_j, this is slowed to the machine speed, U_w, by the pressure in the forming zone.

Web solidity, $\phi(x,y)$:

$$D(\phi_i)\frac{\partial^2\phi_i}{\partial {y_i}^2} + D'(\phi_i)\left(\frac{\partial\phi_i}{\partial y_i}\right)^2 + J_i(x)h_i(x)\frac{\partial\phi_i}{\partial y_i} = \frac{U_w}{L}{h_i}^2(x)\frac{\partial\phi_i}{\partial x} \tag{3}$$

$$J_i(x)h_i(x) = -\left[\frac{D(\phi_i)}{\phi_i}\frac{\partial \phi_i}{\partial y_i}\right]_{y_i=0} \tag{4}$$

$$f(\phi_i(x,0)) = \sigma_i(x)\ ,\ \phi_i(x,1) = \phi_1 \tag{5}$$

where y_i is the dimensionless position through the thickness of the sheet forming on fabric i, scaled by the sheet thickness $h_i(x)$. $J_i(x)$ is the local volumetric flux of fluid leaving each fabric. At the boundary of the sheet and stock the sheet solidity is assumed to be that of the stock suspension, ϕ_1.

Average solidity, $\tilde{\phi}(x)$:

$$\tilde{\phi}_i(x) = \int_0^1 \phi_i dy_i \tag{6}$$

Sheet thickness, $h(x)$:

$$\frac{d}{dx}(\tilde{\phi}_i{}^2(x)h_i{}^2(x)) = -\frac{2LR_t\phi_1\tilde{\phi}_i(x)}{U_w}\left[\frac{D(\phi_i)}{\phi_i}\frac{\partial \phi_i}{\partial y_i}\right]_{y_i=0} \tag{7}$$

$$h_i(x=0) = 0 \tag{8}$$

where R_t is the retention coefficient used by Martinez.

Total flow rate, F_i:

$$F_i = LW\int_0^1 J_i(x)dx \tag{9}$$

where

$$\sigma_i(x) = P_h(x) + (\delta_{i_1} - \delta_{i_2})\frac{1}{2}\frac{\rho {U_w}^2}{R}G(x) - P_{a_i} \tag{10}$$

$$D(\phi_i) = \frac{k(\phi_i)f'(\phi_i)}{\mu}\phi_i \tag{11}$$

$$k(\phi_i) = \frac{1}{3.5{\phi_i}^b {S_0}^2} \tag{12}$$

$$P_s(\phi_i) = f(\phi_i) = m{\phi_i}^n \tag{13}$$

P_h, P_{a_i} and P_s are the measured, ambient and structural pressures respectively. The stock suspension has density ρ, viscosity μ and contains fibres with specific surface S_0. The parameters b, m, and n are constants to be determined by permeability and compressibility experiments.

Martinez assumed that $J_i(x) >> \frac{U_w h_i(x)}{L}$ in order that (3) could be considered an ordinary rather than partial differential equation. However, in the discussion of the model limitations it was noted that this assumption is violated since $J_i(x)$ is only one order of magnitude greater than $\frac{U_w h_i(x)}{L}$.

Here we make the same simplification of (3), but with the correct reasoning. This is that Equation (3) can be considered to be an ordinary differential equation in $\phi_i(y_i)$, rather than a partial differential equation in $\phi_i(x, y_i)$, if and only if $\left|\frac{\partial \phi_i}{\partial x}\right| \approx 0$. From (13) we have

$$\left|\frac{\partial \phi_i}{\partial x}\right| \approx 1 \;\Leftrightarrow\; \left|\frac{\partial P}{\partial x}\right| \approx 1 \tag{14}$$

Hence by considering ϕ as a function of y only, as in [9], the model is limited to the case of constant or near constant pressure in the machine direction.[1]

It is often considered in roll dewatering models that the pressure at the outer fabric is given by the simple formula from membrane theory

$$P_1(y=0) \;=\; \frac{T}{R} \tag{15}$$

The pressure at the inner fabric is less than at the outer fabric because of centrifugal effects [1], this is compensated for by the vacuum in the roll

$$P_2(y=0) \;=\; \frac{T}{R} - \frac{\rho {U_w}^2}{R} G(x) - P_{vac} \tag{16}$$

Since the Martinez model is limited to the case of constant pressure dewatering, we must consider $G(x)$ to be constant in the definition of the boundary condition. We shall see, by comparison with the numerical solution of Martinez [9] for the variable pressure case, that this assumption is valid.

Under the assumption of constant pressure we combine (3) and (4) yielding

$${\phi_i}^{n-b} \frac{d^2 \phi_i}{d{y_i}^2} + (n-b){\phi_i}^{n-b-1} \left(\frac{d\phi_i}{dy_i}\right)^2 - {\phi_{i_0}}^{n-b-1} {\phi_{i_0}}' \frac{d\phi_i}{dy_i} = 0 \tag{17}$$

$$\phi_i(0) \;=\; \left(\frac{P_i(0)}{m}\right)^{\frac{1}{n}} \;=\; \phi_{i_0} \;,\; \phi_i(1) \;=\; \phi_1 \tag{18}$$

[1]Pressure traces measured by Martinez and others [10] show that the pressure is not constant; modelling of the roll pressure is discussed by Zahrai *et al.* [11]

where ${\phi_{i_0}}'$ denotes $\left.\frac{d\phi_i}{dy_i}\right|_{y_i=0}$. Hence the equations to be solved are Equations (17), (4), (6), (7) and (9).

Analytic solution

Equation (17) was solved numerically by Martinez using Runge Kutta and the shooting method. In this section the analytic solution is presented.

It is instructive first to seek a special case. By choosing $n = b+1$ we simplify (17) to

$$\phi_i \frac{d^2\phi_i}{d{y_i}^2} + \left(\frac{d\phi_i}{dy_i}\right)^2 - {\phi_{i_0}}' \frac{d\phi_i}{dy_i} = 0 \tag{19}$$

which is exact and so may be written

$$\frac{d}{dy_i}\left(\phi_i\left(\frac{d\phi_i}{dy_i} - {\phi_{i_0}}'\right)\right) = 0 \tag{20}$$

Now, since $\left.\frac{d\phi_i}{dy_i}\right|_{y_i=0} = {\phi_{i_0}}'$, the constant of integration is zero. The trivial solution, $\phi_i = 0$, does not satisfy the boundary conditions (18). Hence the solution is linear in y. Applying the boundary conditions determines ${\phi_{i_0}}'$ and the second constant of integration, hence the solution

$$\phi_i(y) = \phi_{i_0} - (\phi_{i_0} - \phi_1)y \tag{21}$$

In fact, for general n and b we can solve (17) in the same way to obtain

$$\phi_i(y) = \left({\phi_{i_0}}^{n-b} - \left({\phi_{i_0}}^{n-b} - {\phi_1}^{n-b}\right)y\right)^{\frac{1}{n-b}} \tag{22}$$

Given this analytic solution it is a simple matter to solve equations (4), (6), (7) and (9) to determine the explicit formulae for average solidity, sheet thickness and flow rate.

Average solidity:

$$\tilde{\phi_i} = \frac{n-b}{n-b+1}\,\frac{{\phi_{i_0}}^{n-b+1} - {\phi_1}^{n-b+1}}{{\phi_{i_0}}^{n-b} - {\phi_1}^{n-b}} \tag{23}$$

Sheet thickness:

$$h_i(x) = \left(\frac{4LR_t mn}{7\mu {S_0}^2 U_w}\,\frac{\phi_1 \left|{\phi_{i_0}}'\right| {\phi_0}^{n-b-1}}{\tilde{\phi_i}}\,x\right)^{\frac{1}{2}} \tag{24}$$

Total flow rate:

$$F_i = \left(\frac{4LU_w W^2 mn}{7\mu {S_0}^2 R_t} \frac{\tilde{\phi}_i \left|{\phi_{i_0}}'\right| {\phi_0}^{n-b-1}}{\phi_1} \right)^{\frac{1}{2}} \tag{25}$$

The expression for the sheet thickness (24) predicts that the sheet builds up in proportion to $\sqrt{x}$. This is in agreement with the result stated by Zahrai *et al.* [11]. This result was used by Zahrai *et al.* to predict the pressure around the roll by combining it with their 2D curved blade model.

Equation (25) is an explicit statement of the dependence of flow rate through each fabric on the modelled machine and furnish parameters. The important distinction between our analytic expressions and the numerical solutions of Martinez [9] is that we assume the boundary values for pressure to be constant. We shall see by comparison with Martinez, that the assumption is valid for calculation of flow rates. We note however that Equation (1), used by Martinez, is highly sensitive to pressure profile; integration of Equation (1) therefore does not yield a consistent gap size profile without precise knowledge of the pressure profile.

Effect of machine settings on flow rate

Martinez conducted machine trials on the FEX machine at STFI. The details of the trials are given in [9] and a comparison is made between the numerical solution and measured flow rate F_i through each fabric. The fit of model to experiment is good. In this section we demonstrate the predictive power and ease of application of the analytic results generated in the previous section.

Analytic expressions for flow rate through each fabric exist for general n but are greatly simplified by setting $n = b+1$. Combining (25), (15) and (16) we deduce explicit expressions for the flow rates (measured in $\mathrm{L\,min^{-1}}$) as functions of the machine trial parameters.[2]

$$F_1(U_w, T, P_{vac}) = 6\times 10^4 \left[\frac{2LU_w W^2 mn}{7\mu {S_0}^2 R_t \phi_1} \left(\left(\frac{T}{mR}\right)^{\frac{2}{n}} - {\phi_1}^2 \right) \right]^{\frac{1}{2}} \tag{26}$$

$$F_2(U_w, T, P_{vac}) = 6\times 10^4 \left[\frac{2LU_w W^2 mn}{7\mu {S_0}^2 R_t \phi_1} \left(\left(\frac{T - \rho {U_w}^2 G - RP_{vac}}{mR}\right)^{\frac{2}{n}} - {\phi_1}^2 \right) \right]^{\frac{1}{2}} \tag{27}$$

The tension, speed and roll vacuum are quoted in [9] for each trial. Figures 1 and 2 show how closely the predictions from (26) and (27) match with the measured flow rates.

[2] In [9] $n = 2.56$, $b = 1.5$ so we can invoke this simplification for comparison with the measured results

The predictive power is at least as good as that of the numerical solutions presented by Martinez despite the use of constant pressure and fabric separation. Martinez commented that trials B, K and Q, when the highest roll vacuum of 10 kPa was applied, gave a poor match between experimental data and numerical solution. This was particularly noticeable for the outer fabric. The experimental results record a reduction in flow through the outer fabric with increased vacuum and this is not predicted by the theory. From the analytic solution, it is clear that the flow rate through the outer fabric is independent of the roll vacuum. This is because one of the model assumptions is that the dewatering rates through each fabric are independent.

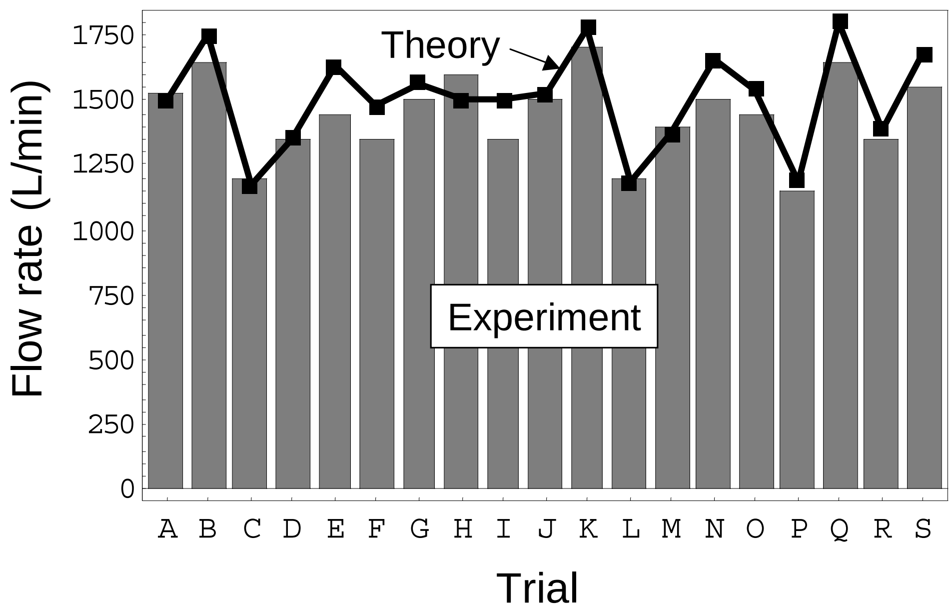

Figure 1: Comparison of analytic solution with measured flow rate through the inner fabric. The fit of the model to experiment is arguably better than that for the numerical solution with measured pressure [9].

Effect of furnish on flow rate

In the previous section we showed how the model can be used to predict the effects of changes in machine conditions. In this section we discuss the potential for using this model to understand the effect of furnish parameters on dewatering.

It is clear from (25) that the flow rate through each fabric is inversely proportional to the fibre specific surface, S_0. This dependence is as expected since increasing S_0 reduces the permeability of the forming web, according to Equation (12). The other two furnish parameters included in the model are the compressibility parameters m and n. Changes in

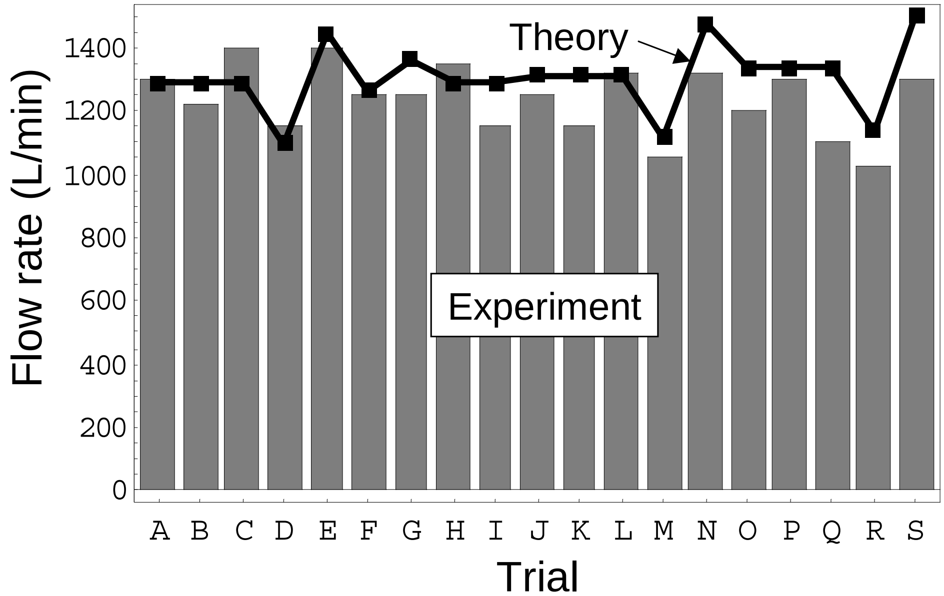

Figure 2: Comparison of analytic solution with measured flow rate through outer fabric. The fit is not as good as for the inner fabric but it is no worse than that given in [9].

m and n also result in a change in the permeability of the forming sheet. The dependence of flow rate through the outer fabric on m and n is shown in Figure 3. The effect of increasing m is to increase the flow rate, but the effect is modest compared to the reduction in flow rate caused by an increase in n.

Conclusions

The analytic solution to the Martinez model of roll dewatering has been presented. This is a significant advance on the previous numerical solution because it is easier to apply and it explicitly states the dependence of flow rate on machine and furnish parameters. The model limitations have been discussed and it has been made clear that both the numerical [9] and analytic solutions are only valid for constant pressure. Hence the measurement of the pressure profile does not improve the predictive power of the model, which is nevertheless very good.

A special, simpler case, of the analytic solution has been identified. This special case is physically relevant because it is the one considered by Martinez in [9]. The simplified form of the analytic solution is used to represent the results from his machine trials, in which machine speed, fabric tension and roll vacuum were varied. The flow rate through the inner fabric is more accurately represented than that through the outer fabric, in

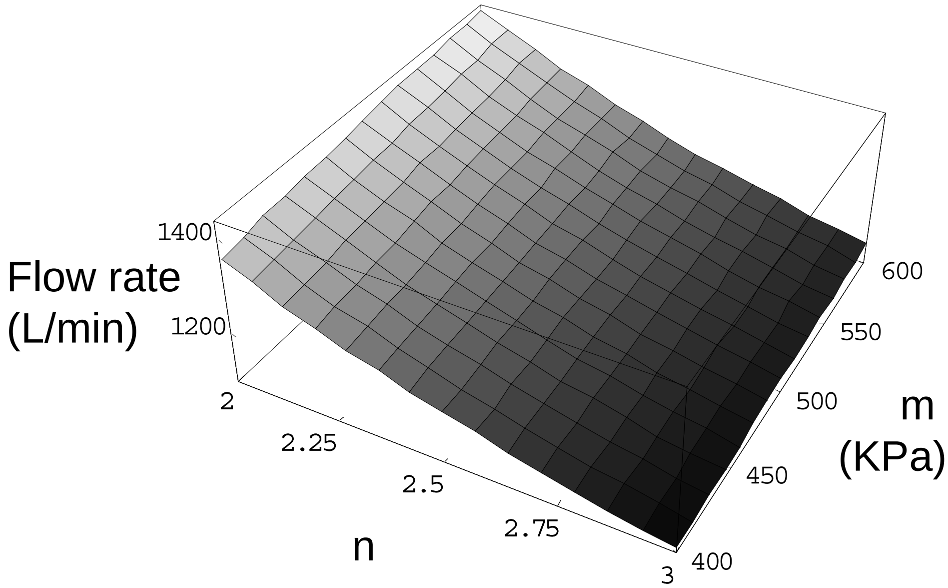

Figure 3: The predicted flow rate through the outer fabric as a function of the compressibility parameters. The flow rate is more sensitive to changes in n than to changes in m. The effect of m is less at higher values of n

agreement with the observations of Martinez [9].

Acknowledgements

The authors acknowledge the support of the Engineering and Physical Sciences Research Council and Shotton Paper Company for financial support of the PhD studies of T. Boxer. Thanks are due also to D. M. Martinez for discussions.

Nomenclature

Symbol	Description	Units
δ_{ij}	Kronecker delta	-
ϕ	web solidity	-
$\tilde{\phi}$	average solidity	-
μ	fluid viscosity	Pa s
ρ	fluid density	$\mathrm{Kg\,m^{-3}}$
σ	hydrodynamic pressure acting on the web surface	Pa
b	permeability constant	-
$D(\phi)$	function defined by equation (11)	$\mathrm{m^2\,s^{-1}}$
$f(\phi)$	compressibility function defined by equation (13)	Pa
F	total volumetric flow rate of fluid leaving the fabric	$\mathrm{m^3\,s^{-1}}$
G	gap size	m
G_0	initial gap size	m
G_L	final gap size	m
h	thickness of the web	m
J	local volumetric flux of fluid leaving the fabrics	$\mathrm{m\,s^{-1}}$
$k(\phi)$	permeability function defined by equation (12)	$\mathrm{m^2}$
L	length of the forming zone	m
m	compressibility constant	Pa
n	compressibility constant	-
P_a	ambient pressure	Pa
P_h	measured pressure trace halfway between fabrics	Pa
P_s	structural pressure supported by the fibre sheet	Pa
R	radius of the roll	m
R_t	retention coefficient	-
S_0	specific surface of the fibre	$\mathrm{m^{-1}}$
T	fabric tension per unit width	$\mathrm{N\,m^{-1}}$
U_w	machine speed	$\mathrm{m\,s^{-1}}$
W	width of fabrics	m
(x, y)	scaled co-ordinates	-

References

[1] Norman, B. "Principles of twin wire forming", *Svesnk Papperstidn.* **82**(11):330-336, (1979).

[2] Zhao, R.H. and Kerekes, R.J. "Pressure distribution between forming fabrics in blade gap formers: thin blades", *J. Pulp Pap. Sci.* **21**(3):J97-J103 (1995).

[3] Zahrai, S. and Bark, F.H. "On the fluid mechanics of twin wire blade forming", *Nord. Pulp Pap. Res. J.* **10**(4):245-252 (1995).

[4] Green, S.I. and Kerekes, R.J. "Numerical analysis of pressure pulses induced by blades in gap formers", *Tappi J.* **81**(4):180-187 (1998).

[5] Nigam, M.S. and Bark, F.H. "An analytic method to calculate the flow past a blade in twin-wire forming", Technical report, Royal Institute of Technology, Stockholm, Sweden, March 1997.

[6] Zahrai S., Bark, F.H. and Norman, B. "An analysis of blade dewatering in a twin-wire paper machine", *J. Pulp Pap. Sci.* **23**(9):J452-J459 (1997).

[7] Green, S.I., Zhao, R.H. and Kerekes, R.J. "Pressure distribution between forming fabrics in blade gap formers: blades of finite width and fabrics of finite stiffness" *J. Pulp Pap. Sci.* **24**(2):J60-J67 (1998).

[8] Green, S.I. "Analytic and computational modelling of twin-wire blade forming", *J. Pulp Pap. Sci.* **23**(7):J353-J357 (1997).

[9] Martinez, D.M. "Characterising the dewatering in roll gap formers", *J. Pulp Pap. Sci.* **24**(1):J7-J13 (1998).

[10] Hegert, R.E. and Sanford, C.L. "Pressure measurements in the forming zone of a twin-wire tissue machine", In *Proc. Int. Water Removal Symp.*, pp47-50, Vancouver, October 1982.

[11] Zahrai, S., Martinez, D.M. and Dahlkild, A.A. "Estimating the tihickness of the web during twin-wire forming", *J. Pulp Pap. Sci.* **24**(2):J67-J72 (1998).