# On the distribution of pore heights in layered random fibre networks

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## Introduction

At a recent research meeting[1], discussion of measurements of internal pore heights from cross sections of paper by K. Niskanen and H. Löytty and of surface topography by M. Lorusso highlighted the need for an appropriate reference model. This short communication fills a gap in the existing theory, which dates back to the work of Corte and Kallmes [1, 2]; it provides the analytic distribution for internal and surface pore heights in a layered random fibre network. Such a network is known to resemble laboratory made paper from dilute suspensions of fibres.

The importance of the development lies in the influence of surface pore structure on printing and fluid entry and the influence of internal void structure on compressibility and hence also printing and converting of paper. We have already a good understanding of the statistical features of horizontal pore structure in paper; now we have a model for the vertical component in random paper. It turns out that the standard deviation of pore height is approximately proportional to the mean pore height.

## Pore height statistics

Two types of pores arise in a random network consisting of $n$ similar layers:

**Surface pores:** vertical sequence of $r$ voids followed by a non-void

**Interior pores:** non-void then vertical sequence of $r$ voids, then non-void.

Evidently the distribution of such heights $r$, measured in units of one fibre thickness (here equivalent to the thickness of one layer) is controlled by the binomial distribution with event probability $p$, the solid fraction in each layer.

[1] EC COST E11, *Geometrical characterisation of paper*, Brussels, 12, 13 November 1998

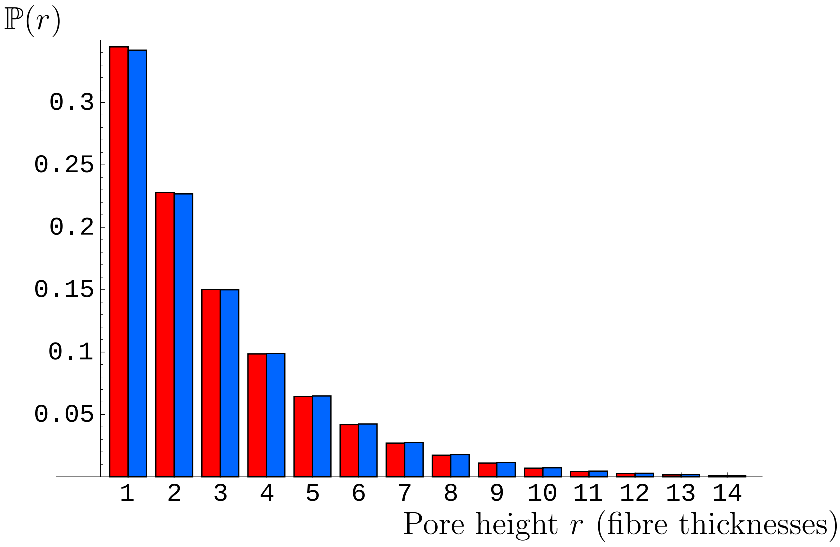


Figure 1: *Probability distribution of internal pore heights (left columns) and surface pore heights (right columns) in units of one fibre thickness for a random sample of* $n = 20$ *layers, each layer having mean solid fraction* $p = 0.3$. *Both have mean pore height about 1.8 units and standard deviation about 2.2 units.*

Recall that the relationship between the mean coverage $c$ in a random fibre layer and the mean solid fraction $p$ in the layer is

$$(1 - p) = e^{-c}. \tag{1}$$

So, for example, in a layer with $p = 0.3$, the mean coverage is $c \approx 0.36$, only 5% of the area has more than 1 fibre coverage and less than 0.6% has more than 2 fibre coverage. The net grammage in a stack of $n = 20$ such layers is typically in the range $40 - 50\,gm^{-2}$; it is given by $np$ times the mean fibre grammage. Analytic expressions for the two distributions of pore height are given in the Appendix. It turns out that both distributions are similar, as may be seen for the examples shown in Figure 1, for a network of $n = 20$ layers with each layer having mean solid fraction $p = 0.3$. The mean pore height is about 1.8 fibre thicknesses, with standard deviation about 2.2 The two main variables are the number of layers $n$, and the solid fraction $p$ in a layer; their effects on the statistics of the distribution of pore heights are illustrated in Figure 2 and Figure 3, respectively. These results seem in agreement with experimental data from cross sections; this and extension of the analysis to non-random networks will be pursued elsewhere.

The interesting point is that for pore height, like many other geometrical features of random networks, here again we find that the standard deviation is for many practical purposes proportional to the mean. For horizontal pore sizes, such a proportionality persists even for *non-random*, flocculated structures; the derivation of the pore height distribution in non-random networks is an open problem for the future.

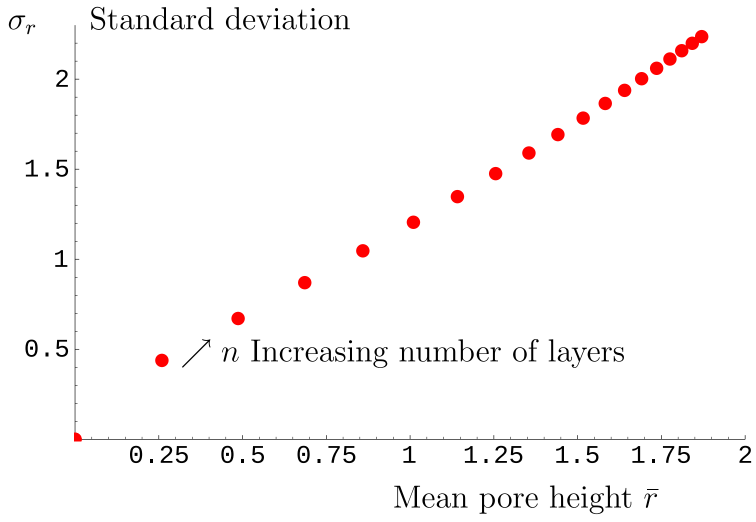


Figure 2: *Effect of number of layers $n$. Plot of standard deviation $\sigma_r$ against mean $\bar{r}$ for pore heights in units of one fibre thickness for random samples of up to $n = 20$ layers, each layer having mean solid fraction $p = 0.3$. The standard deviation is for many practical purposes proportional to the mean. Both internal and surface pores have similar plots; mean and standard deviation increase monotonically with increasing $n$.*

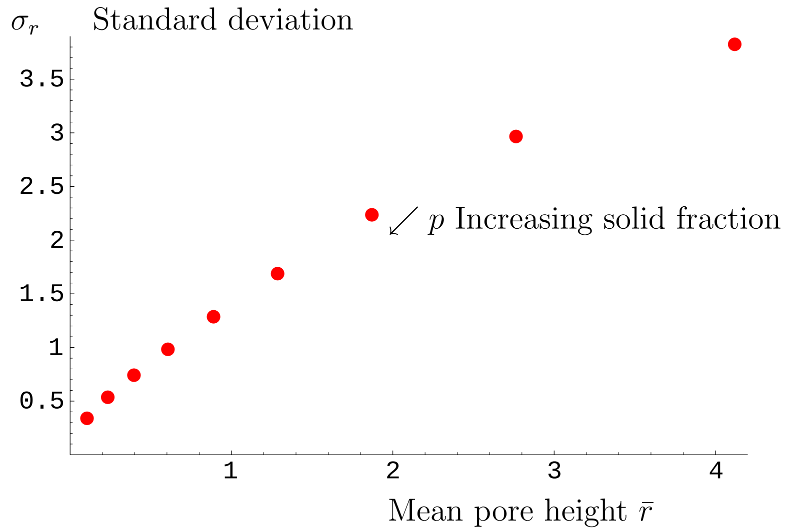


Figure 3: *Effect of layer solid fraction $p$. Plot of standard deviation $\sigma_r$ against mean $\bar{r}$ for pore heights in units of one fibre thickness for random samples of 20 layers, each layer having mean solid fraction $0.1 \leq p \leq 0.9$. The standard deviation is for many practical purposes proportional to the mean. Both internal and surface pores have similar plots; mean and standard deviation decrease monotonically with increasing $p$.*

# Appendix

Consider a stack of $n$ Poisson layers, each with solid fraction $p$. The probability distribution function $\mathbb{P}_{Surf}(r,p,n)$ for **surface pore height** $r$ and its mean $\bar{r}_{Surf}$ are obtained from the binomial distribution and given after some algebraic simplification by

$$\mathbb{P}_{Surf}(r,p,n) = \left(p(1-p)^r(n-r-1)\right) / \left(\sum_{r=0}^{n-1} p(1-p)^r(n-r-1)\right), \quad n = 0,1,\ldots,n-1 \tag{2}$$

$$= \frac{(1-p)^r\, p^2\, (n-r-1)}{-1+(1-p)^n+n\,p} \tag{3}$$

$$\bar{r}_{Surf} = n-2+\frac{2}{p}-\frac{(n-1)\,n\,p}{-1+(1-p)^n+n\,p}. \tag{4}$$

The corresponding probability distribution function $\mathbb{P}_{Int}(r,p,n)$ for **internal pore height** $r$ and its mean $\bar{r}_{Int}$ are obtained similarly and reduce to the following

$$\mathbb{P}_{Int}(r,p,n) = \left(p^2(1-p)^r(n-r-2)\right) / \sum_{r=0}^{n-2}(p^2(1-p)^r(n-r-2)), \quad n = 0,1,\ldots,n-2 \tag{5}$$

$$= \frac{(1-p)^{1+r}\, p^2\, (n-r-2)}{-1+(1-p)^n+p\,(n+p-n\,p)} \tag{6}$$

$$\bar{r}_{Int} = \frac{((1-p)^n\,(2+(-3+n)\,p))+(-1+p)^2\,(-2+(-1+n)\,p)}{p\,(-1+(1-p)^n+p\,(n+p-n\,p))}. \tag{7}$$

The standard deviations $\sigma_{Surf}$ and $\sigma_{Int}$ also are known analytically and available from the author, but their expressions are somewhat cumbersome so they are omitted here.

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# References

[1] H. Corte and O.J. Kallmes. Interpretation of paper properties in terms of structure. In **Formation and structure of paper** Transactions 2nd Fundamental Research Symposium, Oxford 1961. Ed. F. Bolam. BPBMA London 1962 pp.13-46.

[2] O.J. Kallmes and H. Corte. The structure of paper: the statistical geometry of an ideal two-dimensional fibre network *Tappi J.* 43,9 (1960) 737-752. (Errata: *Tappi J.* 44 (1961) 418).