Geometry for Stochastically Inhomogeneous Spacetimes*

C.T.J. Dodson,
Department of Mathematics,
University of Manchester Institute of Science and Technology,
Manchester M60 1QD, UK
dodson@umist.ac.uk

2000

Abstract

Manifolds of gamma probability density functions, which carry a Riemannian information metric, have been used to model the evolution of the intergalactic void statistics and corresponding galactic density statistics. This allowed representation of galaxy clustering using a derived manifold of log-gamma distribution functions to model stochastic spatial variations and provided a metric that measured departures from the random or chaotic state. We show that this log-gamma manifold is an isometric diffeomorph of the gamma manifold and propose a method to approximate matter density statistics in spacetime through a coupling of spacetime and statistical manifold geometries.

1 Introduction

Statistical manifolds [1] are Riemannian manifolds representing the parameter spaces of continuous families of probability density functions. Such spaces have a natural metric induced by the Fisher information matrix which is derived from the gradient of the log-likelihood function, an important function in statistical parameter estimation [3].

Riemannian manifolds of gamma probability density functions have been used to model the evolution of intergalactic void statistics [4] and allowed representation of stochastic matter distributions. This gave a model for galaxy clustering [5], and provided a metric that measured departures from the random or chaotic state. For further details on statistical studies of matter distribution in the universe see for example [6, 13, 18].

Here we consider coupling the spacetime geometry of general relativity with the information geometry of a log-gamma statistical manifold $\mathcal{S}$ that represents spatially stochastic matter distributions. There are polynomial expansions of metric-dependent objects in spacetime local geodesic neighbourhoods that would relate to observable statistics (eg autocovariance gradient of local mean matter densities) in terms of parameters from the log-gamma manifolds. This would be a first step towards a coupled structure which might develop over spacetime a bundle or sheaf of statistical manifolds to represent available stochastic evolutionary features.

2 Statistical manifolds for matter distributions

In [5] the log-gamma family of probability density functions was used to represent spatially stochastic matter distributions for a model of galaxy clustering statistics. This arose from the use [4] of the parent family of gamma distributions for the distribution of intergalactic voids.

Here, we are interested in having distributions of local matter density that include as special cases the uniform distribution, Normal-like distributions, and strongly clustered distributions in

*To appear in Nonlinear Analysis. Presented at WCNA-2000 3^{rd} World Congress of Nonlinear Analysts, Catania, 19-26 July 2000

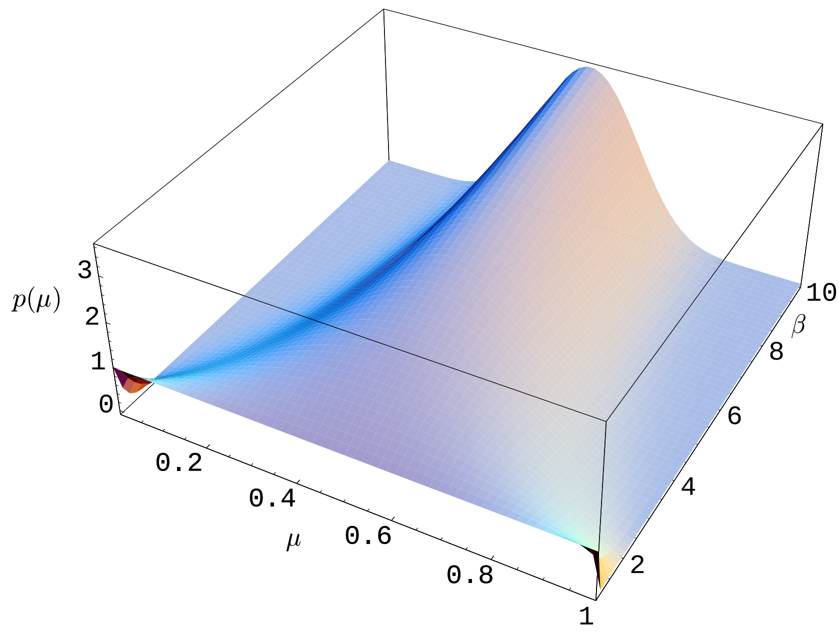

Figure 1: *Part of the surface representing the log-gamma family of densities* $p(\mu; \alpha, \beta)$, *shown for the case of central mean* $< \mu > = \frac{1}{2}$. *Note that the limiting distribution at* $\beta = 1$ *is the uniform distribution and that for larger* β *the distribution resembles a Normal distribution truncated to domain* $[0, 1]$.

spacelike sections of spacetime. We need the random variable representing local matter density μ to be non-negative and bounded above; we take $\mu \in (0, 1]$ and use the log-gamma family, illustrated in Figure 1. Proposition 2.1 establishes the properties we need as a Riemannian 2-manifold.

Proposition 2.1 *The log-gamma family of probability density functions for random variable* $\mu \in (0, 1]$ *given by*

$$p(\mu; \alpha, \beta) = \frac{(\frac{\beta}{\alpha})^{\beta} \left(\log \frac{1}{\mu}\right)^{\beta - 1}}{\mu^{1 - \frac{\beta}{\alpha}}\, \Gamma(\beta)} \qquad \text{for } (\alpha, \beta) \in \mathcal{S} = (0, \infty) \times (0, \infty) \tag{1}$$

determines a statistical 2-manifold of distributions with the following properties
- *it contains the uniform distribution*
- *it contains approximations to truncated Gaussian distributions*
- *the Riemannian structure is given by the information-theoretic metric*
- *it is an isometric diffeomorph of the statistical manifold of gamma distributions.*

Proof
The details are straightforward using the methods of Amari [1]. For the log-gamma densities, the Fisher information matrix determines a Riemannian information metric [1] on the parameter space $\mathcal{S}$. Its arc length function is given by

$$ds^2_{\mathcal{S}} = \frac{\beta}{\alpha^2}\, d\alpha^2 + \left(\psi'(\beta) - \frac{1}{\beta}\right) d\beta^2, \tag{2}$$

where $\psi(\beta) = \frac{\Gamma'(\beta)}{\Gamma(\beta)}$ is the logarithmic derivative of the gamma function, evaluated at β.
In fact, (1) arises from the gamma family of probability density functions

$$q(V; \alpha, \beta) = \frac{V^{\beta - 1} (\frac{\beta}{\alpha})^{\beta}}{\Gamma(\beta)}\, e^{-\frac{V\beta}{\alpha}} \tag{3}$$

for the non-negative random variable $V = \log \frac{1}{\mu}$. It is known that the gamma family (3) has also the information metric (2) (cf [12]) so the identity map on the space of coordinates $\mathcal{S} = \{(\alpha, \beta) \in (0, \infty) \times (0, \infty)\}$ determines an isometry of Riemannian manifolds.

The limiting probability densities are given by

$$\begin{aligned}\lim_{\beta\to 1^+} p(\mu;\alpha,\beta) = p(\mu;\alpha,1) &= \frac{1}{\alpha}\left(\frac{1}{\mu}\right)^{1-\frac{1}{\alpha}} \\ \lim_{\alpha\to 1} p(\mu;\alpha,1) = p(\mu;1,1) &= 1\ .\end{aligned}$$

So $p(\mu;1,1)$ determines the uniform distribution and for larger β the distribution resembles a Normal distribution truncated to domain $[0,1]$, as indicated in Figure 1. $\square$

By direct computation, using *Mathematica* code from Gray [8] we obtain the Gaussian curvature, and other properties.

Corollary 2.2 *The Gaussian curvature of the surface $\mathcal{S}$ is given by:*

$$K_{\mathcal{S}}(\alpha,\beta) = \frac{\psi'(\beta)+\beta\psi''(\beta)}{4(\beta\psi'(\beta)-1} \quad \text{for } \alpha,\beta\in\mathbb{R}^+, \quad \text{where } \psi(\beta) = \frac{\Gamma'(\beta)}{\Gamma(\beta)}$$

$$\begin{aligned}K_{\mathcal{S}}(\alpha,\beta) &\to -\frac{1}{4} \text{ as } \beta\to 0 \\ K_{\mathcal{S}}(\alpha,\beta) &\to -\frac{1}{2} \text{ as } \beta\to\infty\end{aligned}$$ $\square$

Corollary 2.3 *The mean, $<\mu>$, standard deviation σ_μ, and coefficient of variation cv_μ, of μ are given by*

$$\begin{aligned}<\mu> &= \left(\frac{\beta}{\beta+\alpha}\right)^{\beta} \\ \sigma_\mu &= \sqrt{\left(\frac{\beta}{\beta+2\,\alpha}\right)^{\beta} - \left(\frac{\beta}{\beta+\alpha}\right)^{2\,\beta}} \\ cv_\mu &= \frac{\sigma_\mu}{<\mu>} = \sqrt{\left(\frac{\beta}{\beta+2\,\alpha}\right)^{\beta}\left(\frac{\beta+\alpha}{\beta}\right)^{2\,\beta} - 1}.\end{aligned}$$ $\square$

Corollary 2.4 *For the underlying gamma family (3), the information-theoretic entropy [12, 14, 15, 16] is*

$$\begin{aligned}S_q(\alpha,\beta) &= -\int_0^\infty \log(q(x;\alpha,\beta))\,q(x;\alpha,\beta)\,dx \\ &= \beta + (1-\beta)\frac{\Gamma'(\beta)}{\Gamma(\beta)} + \log\frac{\alpha\,\Gamma(\beta)}{\beta}\ .\end{aligned}$$

The maximum entropy occurs at $\beta = 1$, and then $S_q(\alpha,1) = 1+\log\alpha$. $\square$

3 Normal geodesic spacelike neighbourhoods

In this section we collect some well-known local formulae for geometric objects in a Riemannian 3-manifold $(\mathcal{H},h)$ with metric $d\sigma^2 = h_{ij}\,dx^i dx^j$. For further details see eg. [7, 17].

We take an orthonormal frame $(\hat{e}_i)$ at $m\in\mathcal{H}$ and a geodesic γ, with $\gamma(0)=m,\ \gamma'(0)=\hat{u}=\hat{e}_1$. The exponential map is a diffeomorphism in some neighbourhood of $0\in T_m\mathcal{H}$ and suppose that it sends a vector of norm r to $Exp_m(r\hat{u}) = p$. Denote parallel transport of $\hat{e}_i$ along γ by $\hat{E}_i$ and the curvature operator by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X,\nabla_Y].$$

The Jacobi equation is

$$Y'' + R_{\gamma' Y}Y = 0,\quad Y_j(0)=0,\ Y_j'(0)=\hat{e}_j$$

with solution

$$Y_j(r) = r\frac{\partial}{\partial x_j}(\gamma(r)) = (A\hat{E}_j)(r).$$

At $\gamma(r) = p$ we have the following representations in normal coordinates

metric

$$h_{ij}(p) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2}[({}^tAA)(r)] \end{pmatrix}$$

volume density function of Exp_m

$$\theta_m(p) = \frac{1}{r^2}\det A(r).$$

geodesic sphere $G_m(r)$ **of radius** r **centre** m **has** at $\gamma(r) = p$

shape operator $S_m(p)X = (\nabla_X \frac{\partial}{\partial r})(p) = (A'A^{-1})(r)$

mean curvature $H_m(p) = \frac{1}{2}tr(S_m) = \frac{(\det A)'}{\det A}(r) = \frac{1}{r} + \frac{\theta'_m}{2\theta_m}(p)$.

Put $R_m(0) = R_{\hat{u}}\ .\hat{u}$ and $R'_m(0) = (\nabla_{\hat{u}}R)_{\hat{u}}\ .\hat{u}$ and then we have

$$A(r) = rI - \frac{r^3}{6}R_m(0) - \frac{r^4}{12}R'_m(0) + o(r^5)$$

$$A^{-1}(r) = \frac{1}{r}I + \frac{r}{6}R_m(0) + \frac{r^2}{12}R'_m(0) + o(r^3)$$

$$h_{11}(p) = 1;\ \text{for all } j > 1,\ h_{1j}(p) = 0.$$

We denote the Ricci tensor Ric, scalar curvature $Scal$, and Laplacian Δ. For all $i, j = 2, 3$ at $\gamma(r) = p$ we have

$$h_{ij}(p) = \delta_{ij} - \frac{r^2}{3}R_{\hat{u}i\hat{u}j}(m) - \frac{r^3}{6}(\nabla_{\hat{u}}R_{\hat{u}i\hat{u}j})(m) + o(r^4).$$

Volume density function of Exp_m

$$\theta_m(p) = 1 - \frac{r^2}{6}Ric_{\hat{u}\hat{u}}(m) - \frac{r^3}{12}(\nabla_{\hat{u}}Ric_{\hat{u}\hat{u}})(m) + o(r^4).$$

The geodesic sphere $G_m(r)$ has the following geometric objects:

Volume

$$\begin{aligned} V_m(r) &= \pi^{3/2}r^3[1 - \frac{Scal(m)}{30}r^2 + \\ &+ \frac{1}{10500}(-3||R||^2 + 8||Ric||^2 + 5Scal^2 - 18\Delta Scal)(m)r^4 + o(r^6)] \end{aligned}$$

Shape operator

$$S_m(r) = \frac{1}{r}I - \frac{r}{3}R_m(0) - \frac{r^2}{4}R'_m(0) + o(r^3).$$

Mean curvature

$$H_m(r) = \frac{1}{2}tr[S_m(r)] = \frac{1}{r} - \frac{r}{6}Ric_{\hat{u}\hat{u}} - \frac{r^2}{8}(\nabla_{\hat{u}}Ric_{\hat{u}\hat{u}})(m) + o(r^3).$$

4 Spacetime

From Beem et al. [2, 10], Robertson-Walker or Friedmann spacetimes are Lorentzian warped products of the form $\mathcal{M} = \mathbb{R} \times_f \mathcal{H}$ with metric $g = -dt^2 + f(t)\,d\sigma^2$, where $(\mathcal{H}, h)$ is an isotropic Riemannian 3-manifold with $h = d\sigma^2$ and positive warping function $f : \mathbb{R} \to \mathbb{R}^+$. Topologically, $\mathcal{H}$ is one of the spaces $\mathbb{R}^3, \mathbb{S}^3, \mathbb{R}P^3$ with constant sectional curvature of $\mathcal{H}$ rescalable to $0, +1, -1$, respectively. The matter distribution in such spacetimes is homogeneous, $\mu(t, x) =< \mu >$ and this corresponds to the limiting probability distribution function $h(\mu; 1, 1)$ for μ from Proposition 2.1.

Any small perturbation of this state of uniformity would destroy isotropy, so homogeneity is of course physically unstable.

Inhomogeneity can arise in a spherically symmetric but non-isotropic case. This would correspond to spherical matter-shells of differing densities and thickness. The spatial statistics of local matter density observed along a radial line would then be expected to have an autocorrelation function approximated by an exponential decay with decay constant of the order of the mean shell thickness. Spatial statistics of extended objects is discussed in [5].

In spacelike hypersurfaces $\{t\} \times \mathcal{H}$, the representation of stochastic inhomogeneity in the matter distribution would appear through geometric features, which in local geodesic neighbourhoods would have the power series expansions given in the previous section. It is a seemingly impossible task to obtain self-consistent solutions of Einstein's equations incorporating stochasticity in the matter density distribution. On the other hand, it would be desirable to obtain estimates of the statistical rather than deterministic effects of variability in the matter distribution on physically observable quantities—particularly those specific to general relativity. It is in this context that the statistical manifold of probability density functions for local matter density may be deployed.

One way forward is to attempt to develop some statistics of observable geometric features in terms of a perturbation parameter that represents departure from some known deterministic state—such as a uniform matter distribution. Relating back through the available log-gamma family of probability density functions for local matter density would allow estimation of matter-dependent statistics. For example, if we take the case of central mean

$$< \mu >= \left(\frac{\beta}{\beta+\alpha}\right)^{\beta} = \frac{1}{2},$$

then departures from the uniform density state $(\alpha, \beta) = (1,1) \in \mathcal{S}$, would be parametrized by β with α controlled by β through the mean. In this case the perturbation would lie on the curve of constant $< \mu >= \frac{1}{2}$ given by

$$\lambda : [1, \infty) \to \mathcal{S} : \beta \mapsto \left((2^{1/\beta} - 1)\beta, \beta\right). \tag{4}$$

It is proposed to pursue this approach further elsewhere; here we end with mention of one particular area that may be appropriate to investigate.

Hammer [9] analysed the case of a radial beam of light passing through an inhomogeneous spacetime $\mathcal{M}$, consisting of a central core Friedmann model of density μ_c inside radius r_c surrounded by a vacuum shell and both embedded in a background Friedmann spacetime of density $\mu_0 < \mu_c$. Thus, the central core acts as a gravitational lens (cf. also [19]). The behavour of such a system is controlled by the optical scalar equation for the diameter D of the beam as a function of an affine parameter ω along the path of the beam

$$\frac{d^2 D}{d\omega^2} = \frac{1}{2} Ric^{\mathcal{M}}_{ab}\, k^a k^b$$

where k^a represents the wave vector. From the point of view of observable features, it is of interest to solve for the amplification effect of the lensing; this is expressed by D/D_F, the beam diameter relative to a homogeneous Friedmann model. Hammer solved the equations by matching solutions across the boundaries of regions and obtained a power series expansion in terms of the radius r_c of the core dense region. His result gives

$$\frac{D}{D_F} \approx 1 - 6q_0\, H_0 r_c\, (1 - H_c r_c) \left(\frac{\mu_c}{\mu_0} - \left(\frac{\mu_c}{\mu_0}\right)^{1/3} (1+z_c)^2\right)$$

where we have taken the speed of light to be 1; also, H_0, H_c are the Hubble constants for the background and core regions respectively, q_0 is the background deceleration parameter and z_c is the redshift at the core centre. What would be nice would be to have a corresponding result for the central core containing a stochastic distribution of galaxies parametrized by β through Equation (4).

Acknowledgement

The author is grateful to The Royal Society for support with a travel grant.

References

[1] S-I. Amari. **Differential Geometrical Methods in Statistics** Springer Lecture Notes in Statistics 28, Springer-Verlag, Berlin 1985.

[2] J.K. Beem, P.E. Ehrlich and K.L. Easley. **Global Lorentzian Geometry**, Second edition, Marcel Dekker, New York, 1996.

[3] M. Calvo and J.M. Oller. An explicit solution of information geodesic equations for the multivariate normal model. *Statistics & Decisions* 9, (1990) 119-138.

[4] C.T.J. Dodson. Evolution of the void probability function. Presented at **Workshop on Statistics of Cosmological Data Sets**, 8-13 August 1999, Isaac Newton Institute, Cambridge. *Preprint:* `http://www.ma.umist.ac.uk/kd/PREPRINTS/vpf.ps`

[5] C.T.J. Dodson. Spatial statistics and information geometry for parametric statistical models of galaxy clustering. *Int. J. Theor. Phys.*, 38, 10, (1999) 2585-2597.

[6] A.P. Fairall. **Large-Scale Structure In The Universe** Wiley-Praxis, Chichester 1998.

[7] A. Gray. **Tubes**, Addison-Wesley, Redwood City, 1990.

[8] A. Gray. **Modern Differential Geometry of Curves and Surfaces**, Second Edition, CRC Press, Boca Raton, 1998.

[9] F. Hammer. Amplification of light by gravitational lens: Dynamics and thick lens effects. In **Lecture Notes in Physics 212, Proceedings of the conference Gravitation, Geometry and Relativistic Physics**, Aussois 1984, Springer-Verlag, Berlin, 1984 pp 281-285. Cf. also: Gravitational amplification of light: general solution in the vacuole model, *Astron. Astrophys.* 152, 2, 2 (1985) 262-270.

[10] S.W. Hawking and G.F.R. Ellis. **The Large Scale Structure of Space-Time**, Cambridge University Press, Cambridge, 1973.

[11] E.T. Jaynes. Information theory and statistical inference. *The Physical Review* 106 (1957) 620-630 and 108 (1957) 171-190.

[12] S.L. Lauritzen. Statistical Manifolds. In **Differential Geometry in Statistical Inference**, Institute of Mathematical Statistics Lecture Notes, Volume 10, Berkeley 1987, pp 163-218.

[13] P.J.E. Peebles. **Large Scale Structure of the Universe** Princeton University Press, Princeton 1980.

[14] S. Roman. **Coding and Information Theory**, Graduate Texts in Mathematics, 134 Springer-Verlag, New York, 1992, cf. Part 1.

[15] C.E. Shannon. A mathematical theory of communication. *Bell Syst. Tech. J.* 27, (1948) 379-423 and 623-656.

[16] D. Slepian, ed. **Key papers in the development of information theory**, IEEE Press, New York, 1974.

[17] L. Vanhecke. **Geometry in Normal and Tubular Neighbourhoods**, Monograph, 1990, Department of Mathematics, Katholieke Universiteit Leuven, Belgium.

[18] S.D.M. White. The hierarchy of correlation functions and its relation to other measures of galaxy clustering. *Mon. Not. R. Astr. Soc.* 186, (1979) 145-154.

[19] X-P. Wu and F. Hammer. Statistics of lensing by clusters of galaxies. I. Giant arcs. *Mon. Not. R. Astr. Soc.* 262, 1 (1993) 187-203.