Information geometric neighbourhoods of randomness and geometry of the McKay bivariate gamma 3-manifold

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Abstract

We show that gamma distributions provide models for departures from randomness since every neighbourhood of an exponential distribution contains a neighbourhood of gamma distributions, using an information theoretic metric topology. Moreover, every neighbourhood of the uniform distribution contains a neighbourhood of log-gamma distributions. We derive also the information geometry of the 3-manifold of McKay bivariate gamma distributions, which can provide a metrization of departures from randomness and departures from independence for bivariate processes. The curvature objects are derived, including those on three submanifolds. As in the case of bivariate normal manifolds, we have negative scalar curvature but here it is not constant and we show how it depends on correlation. These results have applications, for example, in the characterization of stochastic materials.

1 Gamma distributions and randomness

The family of gamma probability density functions is given by

$$\{p(x;\beta,\alpha) = \left(\frac{\alpha}{\beta}\right)^{\alpha} \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-\frac{\alpha}{\beta}x} | \alpha,\beta \in \mathbb{R}^+\}, \quad x \in \mathbb{R}^+ \tag{1.1}$$

so the space of parameters is topologically $\mathbb{R}^+ \times \mathbb{R}^+$. It is an exponential family and it includes as a special case ($\alpha = 1$) the exponential distribution itself, which complements the Poisson process on a line. It is pertinent to our interests that the property of having sample standard deviation independent of the mean actually characterizes gamma distributions, as shown recently by Hwang and Hu [10]. They proved, for $n \geq 3$ independent positive random variables $x_1, x_2, \ldots, x_n$ with a common continuous probability density function f, that having independence of the sample mean $\bar{x}$ and sample coefficient of variation $cv = S/\bar{x}$ is equivalent to f being a gamma distribution. Of course, the exponential distribution has unit coefficient of variation.

The univariate gamma distribution is widely used to model processes involving a continuous positive random variable. Its information geometry is known and has been applied recently to represent and metrize departures from randomness of, for example, the processes that allocate gaps between occurrences of each amino acid along a protein chain within the *Saccharomyces cerevisiae* genome, see Cai et al [4], clustering of galaxies and communications, Dodson [5, 7, 6].

For comparison purposes, we recall that the information geometry of the univariate and multivariate normal distributions also are known, see for example Lauritzen [11] in Amari et al [1] for the univariate case, Sato et al [13] for the bivariate case and Skovgaard [14] for the multivariate case. We note in particular that the univariate and bivariate normal distributions have constant negative scalar curvature, so geometrically they constitute parts of pseudospheres.

First, we make precise the statement that around every random (Poisson) process there is a neighbourhood of processes governed by the gamma distribution, so gamma distributions can approximate any small enough departure from randomness.

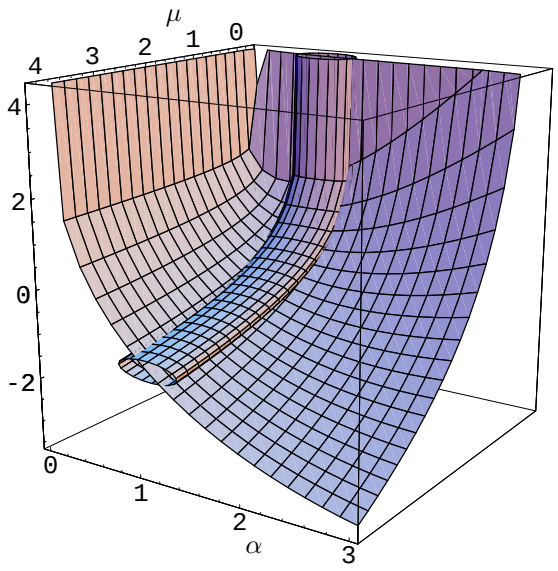

Figure 1: *Affine immersion in natural coordinates $\mu = \alpha/\beta, \alpha$ as a surface in $\mathbb{R}^3$ for the gamma manifold $\mathcal{G}$; the tubular neighbourhood surrounds all exponential distributions—these lie on the curve $\alpha = 1$ in the surface.*

Proposition 1.1 *Every neighbourhood of an exponential distribution contains a neighbourhood of gamma distributions.*

Proof: Dodson and Matsuzoe [8] have provided an affine immersion in Euclidean $\mathbb{R}^3$ for $\mathcal{G}$, the manifold of gamma distributions with Fisher information metric. The coordinates $(\theta^1, \theta^2) = (\mu = \alpha/\beta, \alpha)$ form a natural coordinate system (cf Amari and Nagaoka [2]) for the gamma manifold $\mathcal{G}$. Then $\mathcal{G}$ can be realized in Euclidean $\mathbb{R}^3$ by the graph of the affine immersion $\{h, \xi\}$ where ξ is the transversal vector field along h [2, 8]:

$$h : \mathcal{G} \to \mathbb{R}^3 : \begin{pmatrix} \mu \\ \alpha \end{pmatrix} \mapsto \begin{pmatrix} \mu \\ \alpha \\ \log\Gamma(\alpha) - \alpha \log \mu \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The submanifold of exponential distributions is represented by the curve

$$(0, \infty) \to \mathbb{R}^3 : \mu \mapsto \{\mu, 1, \log \frac{1}{\mu}\}$$

and a tubular neighbourhood of this curve will contain all immersions for small enough perturbations of exponential distributions. In Figure 1 this is depicted in natural coordinates μ, α and in Figure 2 the corresponding surface and tubular neighbourhood (not here an affine immersion, just a continuous image) is shown in the usual (α, β) coordinates of the gamma family (1.1). The tubular neighbourhood in Figure 2 intersects with the gamma manifold immersion to yield the required neighbourhood of gamma distributions, which completes our proof. □

2 Neighbourhoods of the uniform distribution

A simple transformation of random variable $x \in \mathbb{R}^+$ in (1.1) to $z = e^{-x} \in [0, 1]$ converts a gamma distribution into a log-gamma distribution, which turns out to have the same geometry.

Proposition 2.1 (Dodson [7]) *The family of log-gamma probability density functions*

$$\{g(z, \alpha, \beta) = \frac{\frac{1}{z}^{1-\frac{\alpha}{\beta}} (\frac{\alpha}{\beta})^{\alpha} (\log \frac{1}{z})^{\alpha-1}}{\Gamma(\alpha)} | \alpha, \beta \in \mathbb{R}^+\}, \quad z \in [0, 1] \tag{2.2}$$

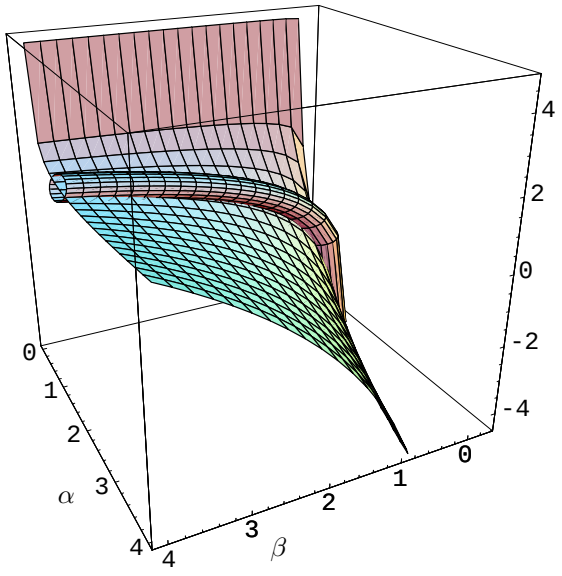

Figure 2: *Continuous image of the affine immersion in Figure 1 as a surface in* $\mathbb{R}^3$ *using standard coordinates for the gamma manifold* $\mathcal{G}$*; the tubular neighbourhood surrounds all exponential distributions—these lie on the curve* $\alpha = 1$ *in the surface.*

determines a Riemannian manifold $\mathcal{L}$ *with information-theoretic metric having the properties:*
- $\mathcal{L}$ *contains the uniform distribution as the limit:* $\lim_{\beta\to 1} g(z,\beta,1) = g(z,1,1) = 1$
- $\mathcal{L}$ *contains approximations to truncated normal distributions for* $\beta >> 1$
- $\mathcal{L}$ *is isometrically equivalent to the gamma manifold* $\mathcal{G}$*.* □

Through this isometry and the result of Dodson and Matsuzoe [8] we have an immersion in $\mathbb{R}^3$ that represents also the log-gamma manifold and, since the isometry sends the exponential distribution to the uniform distribution on $[0,1]$, we obtain another corollary:

Proposition 2.2 *Every neighbourhood of the uniform distribution on* $[0,1]$ *contains a neighbourhood of log-gamma distributions.* □

The value of such topological results lies in the fact that they have qualitative consequences that are stable under small perturbations of a process, something that would be important in real applications. It gives confidence in the use of gamma distributions to model near random processes cf. eg. [4] and log-gamma distributions to model processes subordinate to near-uniform distributions cf. eg. [7]. Moreover, the fact that we have an information-theoretic metric topology for these neighbourhoods lends significance to the result.

3 McKay bivariate gamma distribution

First consider the 3-parameter family of univariate gamma distributions with density function :

$$g(x;\beta,\alpha,\gamma) = \left(\frac{\alpha}{\beta}\right)^{\alpha} \frac{(x-\gamma)^{\alpha-1}}{\Gamma(\alpha)} e^{-\frac{\alpha\,(x-\gamma)}{\beta}}, \quad x > \gamma \geq 0, \quad \beta, \alpha > 0. \tag{3.3}$$

Evidently, the extra parameter $\gamma \geq 0$ is a location shift and when $\gamma = 0$ we recover the univariate gamma distribution (1.1), while when $\alpha = 1$ we obtain the exponential distributions with two parameters. The mean $\bar{x}$, standard deviation σ_x, and coefficient of variation $c\nu_x$, for (3.3) are given by

$$\bar{x} = \beta + \gamma, \quad \sigma_x^2 = \frac{\beta^2}{\alpha}, \quad c\nu_x = \frac{\beta}{\sqrt{\alpha}(\beta+\gamma)}.$$

The distribution (3.3) gives us a slight generalisation of the gamma distribution (1.1), which we use in the sequel.

One of the earliest forms of the bivariate gamma distribution is due to Mckay [12], defined by the density function

$$f(x,y) = \frac{c^{(\alpha_1+\alpha_2)}x^{\alpha_1-1}(y-x)^{\alpha_2-1}e^{-cy}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \text{ defined on } y > x > 0\ ,\ \alpha_1, c, \alpha_2 > 0. \tag{3.4}$$

The marginal distributions are gamma with shape parameters α_1 and $\alpha_1 + \alpha_2$, respectively. The covariance Cov and correlation coefficient ρ of X and Y are given by :

$$Cov(X,Y) = \frac{\alpha_1}{c^2} \tag{3.5}$$

$$\rho(X,Y) = \sqrt{\frac{\alpha_1}{\alpha_1+\alpha_2}}. \tag{3.6}$$

4 Bivariate 3-parameter gamma 5-manifold

In this section we introduce a bivariate gamma distribution which is a slight generalization of that due to Mckay, by substituting $(x-\gamma_1)$ for x and $(y-\gamma_2)$ for y in equation (3.4). We call this a bivariate 3-parameter gamma distribution, because the marginal functions are univariate 3-parameter gamma distributions. Then we consider the bivariate 3-parameter gamma models as a Riemannian 5-manifold. The Christoffel symbols have been calculated but are not listed because they are somewhat cumbersome.

Proposition 4.1 *Let X and Y be continuous random variables, then*

$$f(x,y) = \frac{c^{(\alpha_1+\alpha_2)}(x-\gamma_1)^{\alpha_1-1}(y-\gamma_2-x+\gamma_1)^{\alpha_2-1}e^{-c(y-\gamma_2)}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \tag{4.7}$$

defined on $(y-\gamma_2) > (x-\gamma_1) > 0$, $\alpha_1, c, \alpha_2 > 0$, $\gamma_1, \gamma_2 \geq 0$, is a density function. The covariance and marginal density functions, of X and Y are given by:

$$\sigma_{12} = \frac{\alpha_1}{c^2} \tag{4.8}$$

$$f_X(x) = \frac{c^{\alpha_1}(x-\gamma_1)^{\alpha_1-1}e^{-c(x-\gamma_1)}}{\Gamma(\alpha_1)}, \quad x > \gamma_1 \geq 0 \tag{4.9}$$

$$f_Y(y) = \frac{c^{(\alpha_1+\alpha_2)}(y-\gamma_2)^{(\alpha_1+\alpha_2)-1}e^{-c(y-\gamma_2)}}{\Gamma(\alpha_1+\alpha_2)}, \quad y > \gamma_2 \geq 0. \tag{4.10}$$

□

Note that the marginal functions f_X and f_Y are univariate 3-parameter gamma distributions with parameters (c, α_1, γ_1) and $(c, \alpha_1+\alpha_2, \gamma_2)$, where γ_1 and γ_2 are location parameters. We shall refer to (4.7) as giving the bivariate 3-parameter gamma distributions.

It is easily shown that $c = \sqrt{\frac{\alpha_1}{\sigma_{12}}}$, so the bivariate 3-parameter gamma distribution (4.7) can be presented in the form:

$$f(x,y) = \left(\frac{\alpha_1}{\sigma_{12}}\right)^{\frac{\alpha_1+\alpha_2}{2}} \frac{(x-\gamma_1)^{\alpha_1-1}\,(y-\gamma_2+x-\gamma_1)^{\alpha_2-1}}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)}\, e^{-\sqrt{\frac{\alpha_1}{\sigma_{12}}}\,(y-\gamma_2)}, \tag{4.11}$$

defined on $(y-\gamma_2) > (x-\gamma_1) > 0,$ with parameters $\alpha_1, \alpha_2, \sigma_{12} > 0,$ and $\gamma_1, \gamma_2 \geq 0.$

Proposition 4.2 *Let M^* be the set of bivariate 3-parameter gamma distributions, that is*

$$M^* = \{f | f(x,y) = \left(\frac{\alpha_1}{\sigma_{12}}\right)^{\frac{\alpha_1+\alpha_2}{2}} \frac{(x-\gamma_1)^{\alpha_1-1}\,(y-\gamma_2+x-\gamma_1)^{\alpha_2-1}}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)}\, e^{-\sqrt{\frac{\alpha_1}{\sigma_{12}}}\,(y-\gamma_2)},$$
$$(y-\gamma_2) > (x-\gamma_1) > 0,\ \alpha_1, \alpha_2 > 2,\ \sigma_{12} > 0,\ \gamma_1, \gamma_2 \geq 0.\} \tag{4.12}$$

Then we have :

1. *Identifying* $(\alpha_1, \alpha_2, \sigma_{12}, \gamma_1, \gamma_2)$ *as a local coordinate system,* M^* *can be regarded as a 5-manifold.*

2. M^* *is a Riemannian manifold with Fisher information matrix* $G = [g_{ij}]$ *where*

$$g_{ij} = \int_{\gamma_1}^{\infty} \int_{x-\gamma_1+\gamma_2}^{\infty} \frac{\partial^2 \log f(x,y;x^1,x^2,x^3,x^4,x^5)}{\partial x^i \partial x^j} \; f(x,y;x^1,x^2,x^3,x^4,x^5)\; dy\; dx$$

and $x^1 = \alpha_1$, $x^2 = \alpha_2$, $x^3 = \sigma_{12}$, $x^4 = \gamma_1$, $x^5 = \gamma_2$.
is given by :

$$\begin{bmatrix} \psi'(\alpha_1) + \frac{-3\,\alpha_1+\alpha_2}{4\,\alpha_1{}^2} & \frac{-1}{2\,\alpha_1} & \frac{\alpha_1-\alpha_2}{4\,\alpha_1\,\sigma_{12}} & \frac{\sqrt{\alpha_1}}{(-1+\alpha_1)\,\sqrt{\sigma_{12}}} & \frac{-1}{2\,\sqrt{\alpha_1}\,\sqrt{\sigma_{12}}} \\ \frac{-1}{2\,\alpha_1} & \psi'(\alpha_2) & \frac{1}{2\,\sigma_{12}} & \frac{\sqrt{\alpha_1}}{(1-\alpha_2)\,\sqrt{\sigma_{12}}} & \frac{\sqrt{\alpha_1}}{(-1+\alpha_2)\,\sqrt{\sigma_{12}}} \\ \frac{\alpha_1-\alpha_2}{4\,\alpha_1\,\sigma_{12}} & \frac{1}{2\,\sigma_{12}} & \frac{\alpha_1+\alpha_2}{4\,\sigma_{12}{}^2} & 0 & \frac{\sqrt{\alpha_1}}{2\,\sigma_{12}{}^{\frac{3}{2}}} \\ \frac{\sqrt{\alpha_1}}{(-1+\alpha_1)\,\sqrt{\sigma_{12}}} & \frac{\sqrt{\alpha_1}}{(1-\alpha_2)\,\sqrt{\sigma_{12}}} & 0 & \frac{\alpha_1}{(-2+\alpha_1)\,\sigma_{12}} + \frac{\alpha_1}{(-2+\alpha_2)\,\sigma_{12}} & \frac{-\alpha_1}{(-2+\alpha_2)\,\sigma_{12}} \\ \frac{-1}{2\,\sqrt{\alpha_1}\,\sqrt{\sigma_{12}}} & \frac{\sqrt{\alpha_1}}{(-1+\alpha_2)\,\sqrt{\sigma_{12}}} & \frac{\sqrt{\alpha_1}}{2\,\sigma_{12}{}^{\frac{3}{2}}} & \frac{-\alpha_1}{(-2+\alpha_2)\,\sigma_{12}} & \frac{\alpha_1}{(-2+\alpha_2)\,\sigma_{12}} \end{bmatrix} \tag{4.13}$$

where $\psi(\alpha_i) = \frac{\Gamma'(\alpha_i)}{\Gamma(\alpha_i)} \quad (i = 1, 2).$ □

5 Mckay bivariate gamma 3-manifold

The classical Mckay bivariate gamma distribution becomes a Riemannian 3-manifold, equipped with Fisher information metric. We derive the induced geometry, i.e., the Ricci tensor, the scalar curvatures etc; the Christoffel symbols were computed but are omitted here. In addition, we consider three submanifolds as special cases, and discuss their geometrical structure.

5.0.1 Fisher information metric

The classical family of Mckay bivariate gamma distributions is given by:

$$f(x,y;\alpha_1,\sigma_{12},\alpha_2) = \frac{(\frac{\alpha_1}{\sigma_{12}})^{\frac{(\alpha_1+\alpha_2)}{2}} x^{\alpha_1-1}(y-x)^{\alpha_2-1} e^{-\sqrt{\frac{\alpha_1}{\sigma_{12}}}y}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\;, \tag{5.14}$$

defined on $0 < x < y < \infty$ with parameters $\alpha_1, \sigma_{12}, \alpha_2 > 0$. Where σ_{12} is the covariance of X and Y. The correlation coefficient and marginal functions, of X and Y are given by :

$$\rho(X,Y) = \sqrt{\frac{\alpha_1}{\alpha_1+\alpha_2}} \tag{5.15}$$

$$f_X(x) = \frac{(\frac{\alpha_1}{\sigma_{12}})^{\frac{\alpha_1}{2}} x^{\alpha_1-1} e^{-\sqrt{\frac{\alpha_1}{\sigma_{12}}}x}}{\Gamma(\alpha_1)}, \quad x > 0 \tag{5.16}$$

$$f_Y(y) = \frac{(\frac{\alpha_1}{\sigma_{12}})^{\frac{(\alpha_1+\alpha_2)}{2}} y^{(\alpha_1+\alpha_2)-1} e^{-\sqrt{\frac{\alpha_1}{\sigma_{12}}}y}}{\Gamma(\alpha_1+\alpha_2)}, \quad y > 0. \tag{5.17}$$

Note that it is not possible to choose parameters such that both marginal functions are exponential.

Proposition 5.1 *Let* M *be the set of Mckay bivariate gamma distributions, that is*

$$M = \{f | f(x,y;\alpha_1,\sigma_{12},\alpha_2) = \frac{(\frac{\alpha_1}{\sigma_{12}})^{\frac{(\alpha_1+\alpha_2)}{2}} x^{\alpha_1-1}(y-x)^{\alpha_2-1} e^{-\sqrt{\frac{\alpha_1}{\sigma_{12}}}y}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\,, \\ y > x > 0,\ \alpha_1, \sigma_{12}, \alpha_2 > 0\}. \tag{5.18}$$

Then we have :

1. *Identifying* $(\alpha_1, \sigma_{12}, \alpha_2)$ *as a local coordinate system,* M *is a 3-manifold.*

2. M is a Riemannian 3-manifold with Fisher information metric $G = [g_{ij}]$ given by:

$$[g_{ij}] = \begin{bmatrix} \frac{-3\,\alpha_1+\alpha_2}{4\,{\alpha_1}^2} + \psi'(\alpha_1) & \frac{\alpha_1-\alpha_2}{4\,\alpha_1\,\sigma_{12}} & -\frac{1}{2\,\alpha_1} \\ \frac{\alpha_1-\alpha_2}{4\,\alpha_1\,\sigma_{12}} & \frac{\alpha_1+\alpha_2}{4\,{\sigma_{12}}^2} & \frac{1}{2\,\sigma_{12}} \\ -\frac{1}{2\,\alpha_1} & \frac{1}{2\,\sigma_{12}} & \psi'(\alpha_2) \end{bmatrix} . \tag{5.19}$$

3. The inverse $[g^{ij}]$ of $[g_{ij}]$ is given by:

$$\begin{aligned}
g^{11} &= -\left(\frac{-1+(\alpha_1+\alpha_2)\,\psi'(\alpha_2)}{\psi'(\alpha_2)+\psi'(\alpha_1)\,(1-(\alpha_1+\alpha_2)\,\psi'(\alpha_2))}\right), \\
g^{12} &= g^{21} = \frac{\sigma_{12}\,(1+(\alpha_1-\alpha_2)\,\psi'(\alpha_2))}{\alpha_1\,(\psi'(\alpha_2)+\psi'(\alpha_1)\,(1-(\alpha_1+\alpha_2)\,\psi'(\alpha_2)))}, \\
g^{13} &= g^{31} = \frac{1}{-\psi'(\alpha_2)+\psi'(\alpha_1)\,(-1+(\alpha_1+\alpha_2)\,\psi'(\alpha_2))}, \\
g^{22} &= \frac{{\sigma_{12}}^2\,\left(-1+\left(-3\,\alpha_1+\alpha_2+4\,{\alpha_1}^2\,\psi'(\alpha_1)\right)\,\psi'(\alpha_2)\right)}{{\alpha_1}^2\,(-\psi'(\alpha_2)+\psi'(\alpha_1)\,(-1+(\alpha_1+\alpha_2)\,\psi'(\alpha_2)))}, \\
g^{23} &= g^{32} = \frac{\sigma_{12}\,(-1+2\,\alpha_1\,\psi'(\alpha_1))}{\alpha_1\,(\psi'(\alpha_2)+\psi'(\alpha_1)\,(1-(\alpha_1+\alpha_2)\,\psi'(\alpha_2)))}, \\
g^{33} &= -\left(\frac{-1+(\alpha_1+\alpha_2)\,\psi'(\alpha_1)}{\psi'(\alpha_2)+\psi'(\alpha_1)\,(1-(\alpha_1+\alpha_2)\,\psi'(\alpha_2))}\right) .
\end{aligned} \tag{5.20}$$

□

5.0.2 Curvature properties

We provide the various curvature objects of the McKay 3-manifold M; the Christoffel symbols are known but they are omitted here.

Proposition 5.2 *The components of the curvature tensor R_{ijkl} are given by:*

$$\begin{aligned}
R_{1212} &= \frac{\psi'(\alpha_2)\,(\psi'(\alpha_1)+(\alpha_1+\alpha_2)\,\psi''(\alpha_1))}{16\,{\sigma_{12}}^2\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\,\psi'(\alpha_1)\,\psi'(\alpha_2))}, \\
R_{1213} &= \frac{\psi'(\alpha_2)\,(\psi'(\alpha_1)+2\,\alpha_1\,\psi''(\alpha_1))}{16\,\alpha_1\,\sigma_{12}\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\,\psi'(\alpha_1)\,\psi'(\alpha_2))}, \\
R_{1223} &= \frac{-\psi'(\alpha_1)\,\psi'(\alpha_2)}{16\,{\sigma_{12}}^2\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\,\psi'(\alpha_1)\,\psi'(\alpha_2))}, \\
R_{1313} &= \frac{-\left(-\left(\psi'(\alpha_1)\,\psi'(\alpha_2)\right)+\left(\left(3\,\alpha_1-\alpha_2\right)\,\psi'(\alpha_1)+4\,{\alpha_1}^2\,\psi''(\alpha_1)\right)\,\psi''(\alpha_2)\right)}{16\,{\alpha_1}^2\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\,\psi'(\alpha_1)\,\psi'(\alpha_2))}, \\
R_{1323} &= \frac{-\psi'(\alpha_2)\,(\psi'(\alpha_2)+(-\alpha_1+\alpha_2)\,\psi''(\alpha_2))}{16\,\alpha_1\,\sigma_{12}\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\,\psi'(\alpha_1)\,\psi'(\alpha_2))}, \\
R_{2323} &= \frac{\psi'(\alpha_1)\,(\psi'(\alpha_2)+(\alpha_1+\alpha_2)\,\psi''(\alpha_2))}{16\,{\sigma_{12}}^2\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\,\psi'(\alpha_1)\,\psi'(\alpha_2))}
\end{aligned} \tag{5.21}$$

while the other independent components are zero. □

Proposition 5.3 *The components of the Ricci tensor are given by the symmetric matrix* $R = [R_{ij}]$*:*

$$
\begin{aligned}
R_{11} \;=\;& \frac{-3\,\psi'(\alpha_1)^2\,\psi'(\alpha_2) - 3\,\psi'(\alpha_1)\,\psi'(\alpha_2)^2}{16\,\alpha_1\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \\
& \frac{\alpha_2\,\psi'(\alpha_1)^2\,\psi'(\alpha_2) + \alpha_2\,\psi'(\alpha_1)\,\psi'(\alpha_2)^2}{16\,{\alpha_1}^2\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \\
& \frac{\psi'(\alpha_1)^2\,\psi'(\alpha_2)^2 - 2\,\psi'(\alpha_1)\,\psi'(\alpha_2)\,\psi''(\alpha_1)}{4\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \\
& \frac{-(\alpha_2\,\psi'(\alpha_2)\,\psi''(\alpha_1)) + {\alpha_2}^2\,\psi'(\alpha_2)^2\,\psi''(\alpha_1)}{16\,{\alpha_1}^2\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \\
& \frac{\alpha_1\,\psi'(\alpha_1)\,\psi'(\alpha_2)^2\,\psi''(\alpha_1) + \alpha_2\,\psi'(\alpha_1)\,\psi'(\alpha_2)^2\,\psi''(\alpha_1)}{4\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \\
& \frac{3\,\psi'(\alpha_2)\,\psi''(\alpha_1) + 3\,\psi'(\alpha_1)\,\psi''(\alpha_2)}{16\,\alpha_1\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \\
& \frac{-3\,\psi'(\alpha_2)^2\,\psi''(\alpha_1) - 3\,\psi'(\alpha_1)^2\,\psi''(\alpha_2)}{16\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \\
& \frac{-\left(\alpha_2\,\psi'(\alpha_2)^2\,\psi''(\alpha_1)\right) - \alpha_2\,\psi'(\alpha_1)^2\,\psi''(\alpha_2)}{8\,\alpha_1\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \\
& \frac{-(\alpha_2\,\psi'(\alpha_1)\,\psi''(\alpha_2)) + {\alpha_2}^2\,\psi'(\alpha_1)^2\,\psi''(\alpha_2)}{16\,{\alpha_1}^2\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \\
& \frac{\psi''(\alpha_1)\,\psi''(\alpha_2) - \alpha_1\,\psi'(\alpha_1)\,\psi''(\alpha_1)\,\psi''(\alpha_2) - \alpha_2\,\psi'(\alpha_1)\,\psi''(\alpha_1)\,\psi''(\alpha_2)}{4\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2}
\end{aligned}
$$

$$
\begin{aligned}
R_{12} \;=\;& \frac{\psi'(\alpha_1)^2\,\psi'(\alpha_2) + \psi'(\alpha_1)\,\psi'(\alpha_2)^2 - \psi'(\alpha_2)\,\psi''(\alpha_1)}{16\,\sigma_{12}\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \\
& \frac{-\left(\alpha_2\,\psi'(\alpha_1)^2\,\psi'(\alpha_2)\right) - \alpha_2\,\psi'(\alpha_1)\,\psi'(\alpha_2)^2 + \alpha_2\,\psi'(\alpha_2)\,\psi''(\alpha_1)}{16\,\alpha_1\,\sigma_{12}\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \\
& \frac{\alpha_1\,\psi'(\alpha_2)^2\,\psi''(\alpha_1) - \psi'(\alpha_1)\,\psi''(\alpha_2) + \alpha_1\,\psi'(\alpha_1)^2\,\psi''(\alpha_2)}{16\,\sigma_{12}\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \\
& \frac{-\left({\alpha_2}^2\,\psi'(\alpha_2)^2\,\psi''(\alpha_1)\right) + \alpha_2\,\psi'(\alpha_1)\,\psi''(\alpha_2) - {\alpha_2}^2\,\psi'(\alpha_1)^2\,\psi''(\alpha_2)}{16\,\alpha_1\,\sigma_{12}\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2}
\end{aligned}
$$

$$
\begin{aligned}
R_{13} \;=\;& \frac{-\left(\psi'(\alpha_1)^2\,\psi'(\alpha_2)\right) - \psi'(\alpha_1)\,\psi'(\alpha_2)^2 + \psi'(\alpha_2)\,\psi''(\alpha_1)}{8\,\alpha_1\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \\
& \frac{\psi'(\alpha_2)^2\,\left(\psi''(\alpha_1) + 2\,\psi'(\alpha_1)^2\right) + \psi''(\alpha_2)\left(\psi'(\alpha_1)^2\,+2\,\psi''(\alpha_1)\right)}{8\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \\
& \frac{-\left(\alpha_2\,\psi'(\alpha_2)^2\,\psi''(\alpha_1)\right) + \psi'(\alpha_1)\,\psi''(\alpha_2) - \alpha_2\,\psi'(\alpha_1)^2\,\psi''(\alpha_2)}{8\,\alpha_1\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2},
\end{aligned}
$$

$$
\begin{aligned}
R_{22} \;=\;& \frac{(\alpha_1+\alpha_2)\;(\psi'(\alpha_2)\;(\psi'(\alpha_1)\;(\psi'(\alpha_1)+\psi'(\alpha_2)) - \psi''(\alpha_1) + (\alpha_1+\alpha_2)\;\psi'(\alpha_2)\,\psi''(\alpha_1)))}{16\,{\sigma_{12}}^2\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \\
& \frac{(\alpha_1+\alpha_2)\;(\psi'(\alpha_1)\;(-1+(\alpha_1+\alpha_2)\;\psi'(\alpha_1))\;\psi''(\alpha_2))}{16\,{\sigma_{12}}^2\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\;\psi'(\alpha_1)\,\psi'(\alpha_2))^2},
\end{aligned}
$$

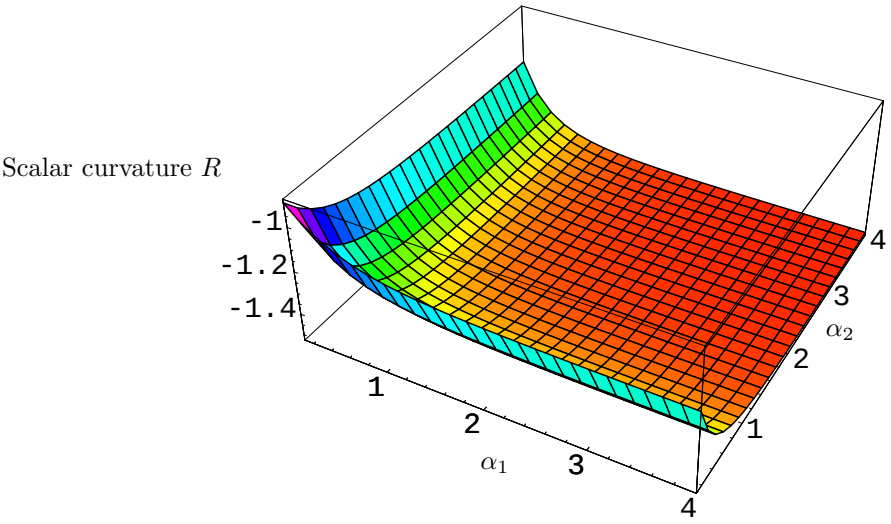

Figure 3: *The scalar curvature R for the McKay bivariate gamma 3-manifold M; the limiting value at the origin is $-\frac{1}{2}$.*

$$R_{23} = \frac{\psi'(\alpha_2)\,(\psi'(\alpha_1)\,(\psi'(\alpha_1)+\psi'(\alpha_2)) - \psi''(\alpha_1) + (\alpha_1+\alpha_2)\,\psi'(\alpha_2)\,\psi''(\alpha_1))}{8\,\sigma_{12}\,(\psi'(\alpha_1)+\psi'(\alpha_2) - (\alpha_1+\alpha_2)\,\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \frac{\psi'(\alpha_1)\,(-1+(\alpha_1+\alpha_2)\,\psi'(\alpha_1))\,\psi''(\alpha_2)}{8\,\sigma_{12}\,(\psi'(\alpha_1)+\psi'(\alpha_2) - (\alpha_1+\alpha_2)\,\psi'(\alpha_1)\,\psi'(\alpha_2))^2},$$

$$R_{33} = \frac{\left(-2\,\psi'(\alpha_1)\,\psi'(\alpha_2) + (\alpha_1+\alpha_2)\,\psi'(\alpha_2)\,\left(\psi'(\alpha_1)^2 - \psi''(\alpha_1)\right) + \psi''(\alpha_1)\right)\,\psi''(\alpha_2)}{4\,(\psi'(\alpha_1)+\psi'(\alpha_2) - (\alpha_1+\alpha_2)\,\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \frac{\psi'(\alpha_1)^2\,\psi'(\alpha_2)^2}{4\,(\psi'(\alpha_1)+\psi'(\alpha_2) - (\alpha_1+\alpha_2)\,\psi'(\alpha_1)\,\psi'(\alpha_2))^2}. \tag{5.22}$$

□

Proposition 5.4 *The scalar curvature R of M is given by:*

$$R = \frac{\psi'(\alpha_1)^2\,\psi'(\alpha_2) + \psi'(\alpha_1)\,\psi'(\alpha_2)^2 + \alpha_1\,\psi'(\alpha_2)^2\,\psi''(\alpha_1)}{2\,(\psi'(\alpha_1)+\psi'(\alpha_2) - (\alpha_1+\alpha_2)\,\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \frac{-(\psi'(\alpha_2)\,\psi''(\alpha_1)) - \psi'(\alpha_1)\,\psi''(\alpha_2)}{(\psi'(\alpha_1)+\psi'(\alpha_2) - (\alpha_1+\alpha_2)\,\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \frac{\alpha_2\,\psi'(\alpha_2)^2\,\psi''(\alpha_1) + \alpha_1\,\psi'(\alpha_1)^2\,\psi''(\alpha_2) + \alpha_2\,\psi'(\alpha_1)^2\,\psi''(\alpha_2)}{2\,(\psi'(\alpha_1)+\psi'(\alpha_2) - (\alpha_1+\alpha_2)\,\psi'(\alpha_1)\,\psi'(\alpha_2))^2} + \frac{-(\alpha_1\,\psi''(\alpha_1)\,\psi''(\alpha_2)) - \alpha_2\,\psi''(\alpha_1)\,\psi''(\alpha_2)}{2\,(\psi'(\alpha_1)+\psi'(\alpha_2) - (\alpha_1+\alpha_2)\,\psi'(\alpha_1)\,\psi'(\alpha_2))^2}. \tag{5.23}$$

This function is negative and has limiting value $-\frac{1}{2}$ as $\alpha_1, \alpha_2 \to 0$. □

Proposition 5.5 *The sectional curvatures of M are given by:*

$$\begin{aligned}
\varrho(1,2) &= \frac{-\psi'(\alpha_2)\ (\psi'(\alpha_1)+(\alpha_1+\alpha_2)\ \psi''(\alpha_1))}{4\ (-1+(\alpha_1+\alpha_2)\ \psi'(\alpha_1))\ (\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\ \psi'(\alpha_1)\,\psi'(\alpha_2))}\,,\\
\varrho(1,3) &= \frac{-\left(\psi'(\alpha_1)\,\psi'(\alpha_2)+\left(\left(-3\,\alpha_1+\alpha_2\right)\ \psi'(\alpha_1)-4\,{\alpha_1}^2\,\psi''(\alpha_1)\right)\ \psi''(\alpha_2)\right)}{4\ (\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\ \psi'(\alpha_1)\,\psi'(\alpha_2))\ (-1+(\alpha_2+\alpha_1\ (-3+4\,\alpha_1\,\psi'(\alpha_1)))\ \psi'(\alpha_2))}\,,\\
\varrho(2,3) &= \frac{-\psi'(\alpha_1)\ (\psi'(\alpha_2)+(\alpha_1+\alpha_2)\ \psi''(\alpha_2))}{4\ (-1+(\alpha_1+\alpha_2)\ \psi'(\alpha_2))\ (\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\ \psi'(\alpha_1)\,\psi'(\alpha_2))}\,. \qquad (5.24)
\end{aligned}$$

□

Proposition 5.6 *The mean curvatures $\varrho(\lambda)\,(\lambda=1,2,3)$ are given by :*

$$\begin{aligned}
\varrho(1) &= \frac{-3\,\alpha_1\,\psi'(\alpha_1)^2\,\psi'(\alpha_2)+\alpha_2\,\psi'(\alpha_1)^2\,\psi'(\alpha_2)-3\,\alpha_1\,\psi'(\alpha_1)\,\psi'(\alpha_2)^2}{8\ (\alpha_2+\alpha_1\ (-3+4\,\alpha_1\,\psi'(\alpha_1)))\ (\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\ \psi'(\alpha_1)\,\psi'(\alpha_2))^2}+\\
&\frac{\alpha_2\,\psi'(\alpha_1)\,\psi'(\alpha_2)^2+4\,{\alpha_1}^2\,\psi'(\alpha_1)^2\,\psi'(\alpha_2)^2+3\,\alpha_1\,\psi'(\alpha_2)\,\psi''(\alpha_1)}{8\ (\alpha_2+\alpha_1\ (-3+4\,\alpha_1\,\psi'(\alpha_1)))\ (\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\ \psi'(\alpha_1)\,\psi'(\alpha_2))^2}+\\
&\frac{-\left(\alpha_2\,\psi'(\alpha_2)\,\psi''(\alpha_1)\right)-8\,{\alpha_1}^2\,\psi'(\alpha_1)\,\psi'(\alpha_2)\,\psi''(\alpha_1)-3\,{\alpha_1}^2\,\psi'(\alpha_2)^2\,\psi''(\alpha_1)}{8\ (\alpha_2+\alpha_1\ (-3+4\,\alpha_1\,\psi'(\alpha_1)))\ (\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\ \psi'(\alpha_1)\,\psi'(\alpha_2))^2}+\\
&\frac{-2\,\alpha_1\,\alpha_2\,\psi'(\alpha_2)^2\,\psi''(\alpha_1)+{\alpha_2}^2\,\psi'(\alpha_2)^2\,\psi''(\alpha_1)+4\,{\alpha_1}^3\,\psi'(\alpha_1)\,\psi'(\alpha_2)^2\,\psi''(\alpha_1)}{8\ (\alpha_2+\alpha_1\ (-3+4\,\alpha_1\,\psi'(\alpha_1)))\ (\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\ \psi'(\alpha_1)\,\psi'(\alpha_2))^2}+\\
&\frac{4\,{\alpha_1}^2\,\alpha_2\,\psi'(\alpha_1)\,\psi'(\alpha_2)^2\,\psi''(\alpha_1)+3\,\alpha_1\,\psi'(\alpha_1)\,\psi''(\alpha_2)-\alpha_2\,\psi'(\alpha_1)\,\psi''(\alpha_2)}{8\ (\alpha_2+\alpha_1\ (-3+4\,\alpha_1\,\psi'(\alpha_1)))\ (\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\ \psi'(\alpha_1)\,\psi'(\alpha_2))^2}+\\
&\frac{-3\,{\alpha_1}^2\,\psi'(\alpha_1)^2\,\psi''(\alpha_2)-2\,\alpha_1\,\alpha_2\,\psi'(\alpha_1)^2\,\psi''(\alpha_2)+{\alpha_2}^2\,\psi'(\alpha_1)^2\,\psi''(\alpha_2)}{8\ (\alpha_2+\alpha_1\ (-3+4\,\alpha_1\,\psi'(\alpha_1)))\ (\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\ \psi'(\alpha_1)\,\psi'(\alpha_2))^2}+\\
&\frac{{\alpha_1}^2\,\psi''(\alpha_1)\,\psi''(\alpha_2)-{\alpha_1}^3\,\psi'(\alpha_1)\,\psi''(\alpha_1)\,\psi''(\alpha_2)-{\alpha_1}^2\,\alpha_2\,\psi'(\alpha_1)\,\psi''(\alpha_1)\,\psi''(\alpha_2)}{2\ (\alpha_2+\alpha_1\ (-3+4\,\alpha_1\,\psi'(\alpha_1)))\ (\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\ \psi'(\alpha_1)\,\psi'(\alpha_2))^2}\,,\\
\varrho(2) &= \frac{\psi'(\alpha_2)\ (\psi'(\alpha_1)\ (\psi'(\alpha_1)+\psi'(\alpha_2))-\psi''(\alpha_1)+(\alpha_1+\alpha_2)\ \psi'(\alpha_2)\,\psi''(\alpha_1))}{8\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\ \psi'(\alpha_1)\,\psi'(\alpha_2))^2}+\\
&\frac{\psi'(\alpha_1)\ (-1+(\alpha_1+\alpha_2)\ \psi'(\alpha_1))\ \psi''(\alpha_2)}{8\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\ \psi'(\alpha_1)\,\psi'(\alpha_2))^2}\,,\\
\varrho(3) &= \frac{\left(-2\,\psi'(\alpha_1)\,\psi'(\alpha_2)+(\alpha_1+\alpha_2)\ \psi'(\alpha_2)\ \left(\psi'(\alpha_1)^2-\psi''(\alpha_1)\right)+\psi''(\alpha_1)\right)\ \psi''(\alpha_2)}{8\,\psi'(\alpha_2)\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\ \psi'(\alpha_1)\,\psi'(\alpha_2))^2}+\\
&\frac{\psi'(\alpha_1)^2\,\psi'(\alpha_2)^2}{8\,\psi'(\alpha_2)\,(\psi'(\alpha_1)+\psi'(\alpha_2)-(\alpha_1+\alpha_2)\ \psi'(\alpha_1)\,\psi'(\alpha_2))^2}\,. \qquad (5.25)
\end{aligned}$$

□

6 Submanifolds of the Mckay 3-manifold M

We consider three submanifolds M_1, M_2 and M_3 of the 3-manifold M of Mckay bivariate gamma distributions (5.18) $f(x,y;\alpha_1,\sigma_{12},\alpha_2)$, where we use the coordinate system $(\alpha_1,\sigma_{12},\alpha_2)$. These submanifolds have dimension 2 and so it follows that the scalar curvature is twice the Gaussian curvature, $R=2K$. Recall from above that the correlation is given by

$$\rho=\sqrt{\frac{\alpha_1}{\alpha_1+\alpha_2}}.$$

In the cases of M_1 and M_2 the scalar curvature can be presented as a function only of ρ, this is shown in Figure 4.

6.1 Submanifold $M_1 \subset M$: $\alpha_1 = 1$

The distributions are of form:

$$f(x,y;1,\sigma_{12},\alpha_2) = \frac{(\frac{1}{\sigma_{12}})^{\frac{1+\alpha_2}{2}}(y-x)^{\alpha_2-1}e^{-\sqrt{\frac{1}{\sigma_{12}}}y}}{\Gamma(\alpha_2)}\ , \tag{6.26}$$

defined on $0 < x < y < \infty$ with parameters $\sigma_{12}, \alpha_2 > 0$. The correlation coefficient and marginal functions, of X and Y are given by :

$$\begin{aligned}
\rho(X,Y) &= \frac{1}{\sqrt{1+\alpha_2}} && (6.27)\\
f_X(x) &= \frac{1}{\sqrt{\sigma_{12}}}\, e^{-\frac{1}{\sqrt{\sigma_{12}}}x}, \quad x>0 && (6.28)\\
f_Y(y) &= \frac{(\frac{1}{\sqrt{\sigma_{12}}})^{(1+\alpha_2)} y^{\alpha_2} e^{-\frac{1}{\sqrt{\sigma_{12}}}y}}{\alpha_2\,\Gamma(\alpha_2)}, \quad y>0\ . && (6.29)
\end{aligned}$$

So here we have $\alpha_2 = \frac{1-\rho^2}{\rho^2}$, which in practice would give a measure of the variability not due to the correlation.

Proposition 6.1 *The metric tensor $[g_{ij}]$ and its inverse $[g^{ij}]$ are as follows :*

$$G = [g_{ij}] = \begin{bmatrix} \frac{1+\alpha_2}{4\,{\sigma_{12}}^2} & \frac{1}{2\,\sigma_{12}} \\ \frac{1}{2\,\sigma_{12}} & \psi'(\alpha_2) \end{bmatrix} \tag{6.30}$$

$$G^{-1} = [g^{ij}] = \begin{bmatrix} \frac{4\,{\sigma_{12}}^2\,\psi'(\alpha_2)}{-1+(1+\alpha_2)\,\psi'(\alpha_2)} & \frac{-2\,\sigma_{12}}{-1+(1+\alpha_2)\,\psi'(\alpha_2)} \\ \frac{-2\,\sigma_{12}}{-1+(1+\alpha_2)\,\psi'(\alpha_2)} & \frac{1+\alpha_2}{-1+(1+\alpha_2)\,\psi'(\alpha_2)} \end{bmatrix}\ . \tag{6.31}$$

□

Proposition 6.2 *The Christoffel symbols of M_1 are*

$$\begin{aligned}
\Gamma^1_{11} &= \frac{-4+\frac{1}{-1+(1+\alpha_2)\,\psi'(\alpha_2)}}{4\,\sigma_{12}}\,,\\
\Gamma^1_{12} &= \Gamma^1_{21} = \frac{\psi'(\alpha_2)}{-2+2\ (1+\alpha_2)\ \psi'(\alpha_2)}\,,\\
\Gamma^1_{22} &= -\left(\frac{\sigma_{12}\,\psi''(\alpha_2)}{-1+(1+\alpha_2)\ \psi'(\alpha_2)}\right),\\
\Gamma^2_{11} &= \frac{-(1+\alpha_2)}{8\,{\sigma_{12}}^2\ (-1+(1+\alpha_2)\ \psi'(\alpha_2))}\,,\\
\Gamma^2_{12} &= \Gamma^2_{21} = \frac{-1}{4\,\sigma_{12}\ (-1+(1+\alpha_2)\ \psi'(\alpha_2))}\,,\\
\Gamma^2_{22} &= \frac{(1+\alpha_2)\ \psi''(\alpha_2)}{-2+2\ (1+\alpha_2)\ \psi'(\alpha_2)}\,.
\end{aligned} \tag{6.32}$$

□

Proposition 6.3 *The curvature tensor of M_1 is given by*

$$R_{1212} = \frac{-(\psi'(\alpha_2)+(1+\alpha_2)\ \psi''(\alpha_2))}{16\,{\sigma_{12}}^2\ (-1+(1+\alpha_2)\ \psi'(\alpha_2))}\,, \tag{6.33}$$

while the other independent components are zero.

By contraction we obtain:

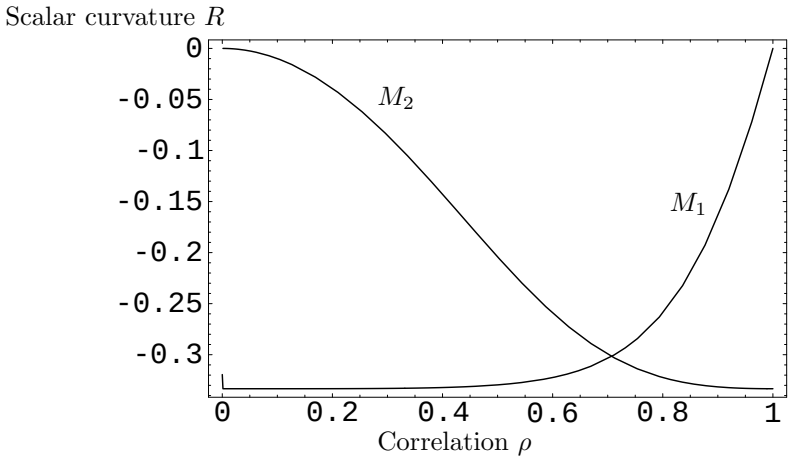

Figure 4: *The scalar curvature R as a function of correlation ρ for McKay submanifolds: M_1 (M with $\alpha_1 = 1$) where R increases from $-\frac{1}{3}$ to 0, and M_2 (M with $\alpha_2 = 1$) where R decreases from 0 to $-\frac{1}{3}$.*

Ricci tensor:

$$\begin{aligned} R_{11} &= \frac{(1+\alpha_2)\ (\psi'(\alpha_2) + (1+\alpha_2)\ \psi''(\alpha_2))}{16\,{\sigma_{12}}^2\,(-1+(1+\alpha_2)\ \psi'(\alpha_2))^2}\,, \\ R_{12} &= \frac{\psi'(\alpha_2) + (1+\alpha_2)\ \psi''(\alpha_2)}{8\,\sigma_{12}\,(-1+(1+\alpha_2)\ \psi'(\alpha_2))^2}\,, \\ R_{22} &= \frac{\psi'(\alpha_2)\ (\psi'(\alpha_2) + (1+\alpha_2)\ \psi''(\alpha_2))}{4\,(-1+(1+\alpha_2)\ \psi'(\alpha_2))^2}\,. \end{aligned} \tag{6.34}$$

Scalar curvature:

$$R = \frac{\psi'(\alpha_2) + (1+\alpha_2)\ \psi''(\alpha_2)}{2\,(-1+(1+\alpha_2)\ \psi'(\alpha_2))^2}\,. \tag{6.35}$$

□

6.2 Submanifold $M_2 \subset M$: $\alpha_2 = 1$

The distributions are of form:

$$f(x,y;\alpha_1,\sigma_{12},1) = \frac{(\frac{\alpha_1}{\sigma_{12}})^{\frac{\alpha_1+1}{2}} x^{\alpha_1-1} e^{-\sqrt{\frac{\alpha_1}{\sigma_{12}}}y}}{\Gamma(\alpha_1)}\,, \tag{6.36}$$

defined on $0 < x < y < \infty$ with parameters $\alpha_1, \sigma_{12} > 0$. The correlation coefficient and marginal functions, of X and Y are given by :

$$\rho(X,Y) = \sqrt{\frac{\alpha_1}{1+\alpha_1}} \tag{6.37}$$

$$f_X(x) = \frac{(\frac{\alpha_1}{\sigma_{12}})^{\frac{\alpha_1}{2}} x^{\alpha_1-1} e^{-\sqrt{\frac{\alpha_1}{\sigma_{12}}}x}}{\Gamma(\alpha_1)}, \quad x > 0 \tag{6.38}$$

$$f_Y(y) = \frac{(\frac{\alpha_1}{\sigma_{12}})^{\frac{(\alpha_1+1)}{2}} y^{\alpha_1} e^{-\sqrt{\frac{\alpha_1}{\sigma_{12}}}y}}{\alpha_1\,\Gamma(\alpha_1)}, \quad y > 0\,. \tag{6.39}$$

Here we have $\alpha_1 = \frac{\rho^2}{1-\rho^2}$.

Proposition 6.4 *The metric tensor* $[g_{ij}]$ *and its inverse* $[g^{ij}]$ *are as follows :*

$$G = [g_{ij}] = \begin{bmatrix} \frac{1-3\,\alpha_1}{4\,{\alpha_1}^2} + \psi'(\alpha_1) & \frac{-1+\alpha_1}{4\,\alpha_1\,\sigma_{12}} \\ \frac{-1+\alpha_1}{4\,\alpha_1\,\sigma_{12}} & \frac{1+\alpha_1}{4\,{\sigma_{12}}^2} \end{bmatrix} . \tag{6.40}$$

$$G^{-1} = [g^{ij}] = \begin{bmatrix} \frac{1+\alpha_1}{-1+(1+\alpha_1)\,\psi'(\alpha_1)} & -\left(\frac{(-1+\alpha_1)\,\sigma_{12}}{\alpha_1\,(-1+(1+\alpha_1)\,\psi'(\alpha_1))}\right) \\ -\left(\frac{(-1+\alpha_1)\,\sigma_{12}}{\alpha_1\,(-1+(1+\alpha_1)\,\psi'(\alpha_1))}\right) & \frac{{\sigma_{12}}^2\left(1+\alpha_1\left(-3+4\,\alpha_1\,\psi'(\alpha_1)\right)\right)}{{\alpha_1}^2\,(-1+(1+\alpha_1)\,\psi'(\alpha_1))} \end{bmatrix} \tag{6.41}$$

□

Proposition 6.5 *The Christoffel symbols are*

$$\begin{aligned} \Gamma^1_{11} &= \frac{-1+\alpha_1\,(3+4\,\alpha_1\,(1+\alpha_1)\,\psi''(\alpha_1))}{8\,{\alpha_1}^2\,(-1+(1+\alpha_1)\,\psi'(\alpha_1))}\,, \\ \Gamma^1_{12} &= \Gamma^1_{21} = \frac{-\,(-1+\alpha_1)}{8\,\alpha_1\,\sigma_{12}\,(-1+(1+\alpha_1)\,\psi'(\alpha_1))}\,, \\ \Gamma^1_{22} &= \frac{-\,(1+\alpha_1)}{8\,{\sigma_{12}}^2\,(-1+(1+\alpha_1)\,\psi'(\alpha_1))}\,, \\ \Gamma^2_{11} &= \frac{\sigma_{12}\,(-1+\alpha_1\,(-3+8\,\psi'(\alpha_1)-4\,(-1+\alpha_1)\,\alpha_1\,\psi''(\alpha_1)))}{8\,{\alpha_1}^3\,(-1+(1+\alpha_1)\,\psi'(\alpha_1))}\,, \\ \Gamma^2_{12} &= \frac{1+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1))}{8\,{\alpha_1}^2\,(-1+(1+\alpha_1)\,\psi'(\alpha_1))}, \\ \Gamma^2_{22} &= \frac{-8+\frac{-1+\alpha_1}{\alpha_1\,(-1+(1+\alpha_1)\,\psi'(\alpha_1))}}{8\,\sigma_{12}}\,. \end{aligned} \tag{6.42}$$

□

Proposition 6.6 *The curvature tensor is given by*

$$R_{1212} = \frac{-\,(\psi'(\alpha_1)+(1+\alpha_1)\,\psi''(\alpha_1))}{16\,{\sigma_{12}}^2\,(-1+(1+\alpha_1)\,\psi'(\alpha_1))}. \tag{6.43}$$

while the other independent components are zero.

By contraction we obtain:
Ricci tensor:

$$\begin{aligned} R_{11} &= \frac{(1+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,(\psi'(\alpha_1)+(1+\alpha_1)\,\psi''(\alpha_1))}{16\,{\alpha_1}^2\,(-1+(1+\alpha_1)\,\psi'(\alpha_1))^2}\,, \\ R_{21} &= \frac{(-1+\alpha_1)\,(\psi'(\alpha_1)+(1+\alpha_1)\,\psi''(\alpha_1))}{16\,\alpha_1\,\sigma_{12}\,(-1+(1+\alpha_1)\,\psi'(\alpha_1))^2}\,, \\ R_{22} &= \frac{(1+\alpha_1)\,(\psi'(\alpha_1)+(1+\alpha_1)\,\psi''(\alpha_1))}{16\,{\sigma_{12}}^2\,(-1+(1+\alpha_1)\,\psi'(\alpha_1))^2}\,. \end{aligned} \tag{6.44}$$

Scalar curvature:

$$R = \frac{\psi'(\alpha_1)+(1+\alpha_1)\,\psi''(\alpha_1)}{2\,(-1+(1+\alpha_1)\,\psi'(\alpha_1))^2}\,. \tag{6.45}$$

□

6.3 Submanifold $M_3 \subset M$: $\sigma_{12} = 1$

Here the distributions have unit covariance and are of form:

$$f(x,y;\alpha_1,1,\alpha_2) = \frac{(\alpha_1)^{\frac{(\alpha_1+\alpha_2)}{2}}\,x^{\alpha_1-1}(y-x)^{\alpha_2-1}e^{-\sqrt{\alpha_1}y}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\,, \tag{6.46}$$

defined on $0 < x < y < \infty$ with parameters $\alpha_1, \alpha_2 > 0$. The correlation coefficient and marginal functions are given by:

$$\begin{aligned}
\rho(X,Y) &= \sqrt{\frac{\alpha_1}{\alpha_1+\alpha_2}} && (6.47)\\
f_X(x) &= \frac{\sqrt{\alpha_1}^{\alpha_1} x^{\alpha_1-1} e^{-\sqrt{\alpha_1}x}}{\Gamma(\alpha_1)}, \quad x>0 && (6.48)\\
f_Y(y) &= \frac{\sqrt{\alpha_1}^{(\alpha_1+\alpha_2)} y^{(\alpha_1+\alpha_2)-1} e^{-\sqrt{\alpha_1}y}}{\Gamma(\alpha_1+\alpha_2)}, \quad y>0\ . && (6.49)
\end{aligned}$$

Proposition 6.7 *The metric tensor $[g_{ij}]$ and its inverse $[g^{ij}]$ are as follows:*

$$G = [g_{ij}] = \begin{bmatrix} \frac{-3\,\alpha_1+\alpha_2}{4\,{\alpha_1}^2} + \psi'(\alpha_1) & \frac{-1}{2\,\alpha_1} \\ \frac{-1}{2\,\alpha_1} & \psi'(\alpha_2) \end{bmatrix} \tag{6.50}$$

$$G^{-1} = [g^{ij}] = \begin{bmatrix} \frac{4\,{\alpha_1}^2\,\psi'(\alpha_2)}{-1+(\alpha_2+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,\psi'(\alpha_2)} & \frac{2\,\alpha_1}{-1+(\alpha_2+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,\psi'(\alpha_2)} \\ \frac{2\,\alpha_1}{-1+(\alpha_2+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,\psi'(\alpha_2)} & \frac{\alpha_2+\alpha_1\left(-3+4\,\alpha_1\,\psi'(\alpha_1)\right)}{-1+(\alpha_2+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,\psi'(\alpha_2)} \end{bmatrix}. \tag{6.51}$$

□

Proposition 6.8 *The Christoffel symbols are*

$$\begin{aligned}
\Gamma^1_{11} &= \frac{3+2\,\psi'(\alpha_2)\,\left(3\,\alpha_1-2\,\alpha_2+4\,{\alpha_1}^3\,\psi''(\alpha_1)\right)}{4\,\alpha_1\,\left(-1+(\alpha_2+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,\psi'(\alpha_2)\right)},\\
\Gamma^1_{22} &= \frac{\alpha_1\,\psi''(\alpha_2)}{-1+(\alpha_2+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,\psi'(\alpha_2)},\\
\Gamma^1_{12} &= \Gamma^1_{21} = \frac{\psi'(\alpha_2)}{-2+2\,(\alpha_2+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,\psi'(\alpha_2)},\\
\Gamma^2_{11} &= \frac{-\alpha_2+\alpha_1\,\left(-3+12\,\alpha_1\,\psi'(\alpha_1)+8\,{\alpha_1}^2\,\psi''(\alpha_1)\right)}{8\,{\alpha_1}^2\,\left(-1+(\alpha_2+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,\psi'(\alpha_2)\right)},\\
\Gamma^2_{22} &= \frac{(\alpha_2+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,\psi''(\alpha_2)}{-2+2\,(\alpha_2+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,\psi'(\alpha_2)},\\
\Gamma^2_{12} &= \Gamma^2_{21} = \frac{1}{4\,\alpha_1\,\left(-1+(\alpha_2+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,\psi'(\alpha_2)\right)}. && (6.52)
\end{aligned}$$

□

Proposition 6.9 *The curvature tensor is given by*

$$R_{1212} = \frac{-\psi'(\alpha_2)+\left(-\alpha_2+\alpha_1\,\left(-3+12\,\alpha_1\,\psi'(\alpha_1)+8\,{\alpha_1}^2\,\psi''(\alpha_1)\right)\right)\,\psi''(\alpha_2)}{16\,{\alpha_1}^2\,\left(-1+(\alpha_2+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,\psi'(\alpha_2)\right)}, \tag{6.53}$$

while the other independent components are zero.

By contraction we obtain:
Ricci tensor:

$$\begin{aligned}
R_{11} &= \frac{(\alpha_2+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,\left(\left(-\alpha_2+\alpha_1\,\left(-3+12\,\alpha_1\,\psi'(\alpha_1)+8\,{\alpha_1}^2\,\psi''(\alpha_1)\right)\right)\,\psi''(\alpha_2)\right)}{-16\,{\alpha_1}^2\,\left(-1+(\alpha_2+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,\psi'(\alpha_2)\right)^2}\\
&\quad+\frac{\psi'(\alpha_2)\,(\alpha_2+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,(\psi'(\alpha_2))}{16\,{\alpha_1}^2\,\left(-1+(\alpha_2+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,\psi'(\alpha_2)\right)^2},\\
R_{12} &= \frac{-\psi'(\alpha_2)+\left(-\alpha_2+\alpha_1\,\left(-3+12\,\alpha_1\,\psi'(\alpha_1)+8\,{\alpha_1}^2\,\psi''(\alpha_1)\right)\right)\,\psi''(\alpha_2)}{8\,\alpha_1\,\left(-1+(\alpha_2+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,\psi'(\alpha_2)\right)^2},\\
R_{22} &= \frac{\psi'(\alpha_2)\,(\psi'(\alpha_2)+(\alpha_2+\alpha_1\,(3-4\,\alpha_1\,(3\,\psi'(\alpha_1)+2\,\alpha_1\,\psi''(\alpha_1))))\,\psi''(\alpha_2))}{4\,\left(-1+(\alpha_2+\alpha_1\,(-3+4\,\alpha_1\,\psi'(\alpha_1)))\,\psi'(\alpha_2)\right)^2}. && (6.54)
\end{aligned}$$

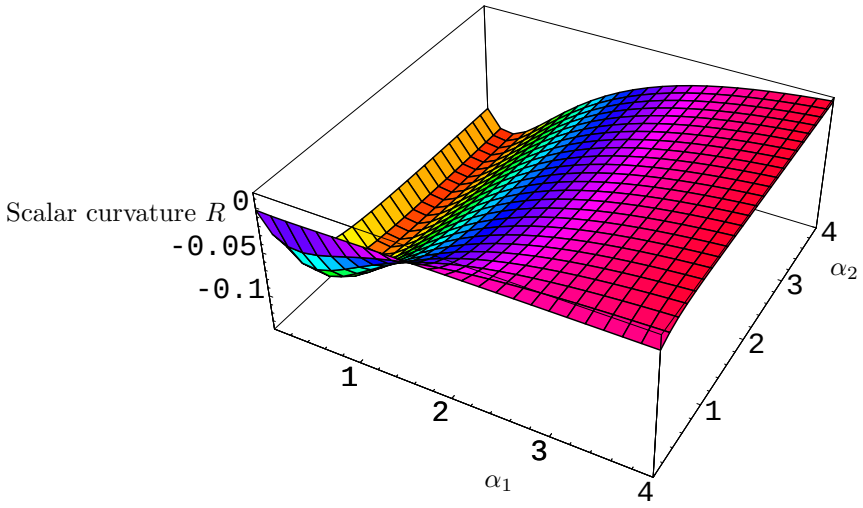

Figure 5: *The scalar curvature R for McKay submanifold M_3, (M with $\sigma_{12} = 1$).*

Scalar curvature:

$$R = \frac{\psi'(\alpha_2) + (\alpha_2 + \alpha_1\ (3 - 4\,\alpha_1\ (3\,\psi'(\alpha_1) + 2\,\alpha_1\,\psi''(\alpha_1))))\ \psi''(\alpha_2)}{2\,(-1 + (\alpha_2 + \alpha_1\ (-3 + 4\,\alpha_1\,\psi'(\alpha_1)))\ \psi'(\alpha_2))^2}\,. \tag{6.55}$$

□

7 Applications

The univariate gamma information geometry is known and has been applied recently to represent and metrize departures from randomness of, for example, the processes that allocate gaps between occurrences of each amino acid along a protein chain within the *Saccharomyces cerevisiae* genome, see Cai et al [4], cosmological void distribution and clustering of galaxies, and communications, Dodson [5, 7, 6].

The new results on bivariate gamma geometry have potential applications in any situation where positively correlated variables $0 < x < y < \infty$ are used to model a process with marginal gamma distributions. In some of the applications mentioned above, there are other variables associated and these may yield some refinements of the existing models. Two new case studies are being developed at present, both involve bivariate data from measurements on stochastic porous media.

The first application concerns the structure of paper and nonwoven textiles, in the manufacture of which fibres are deposited on a continuous filter bed and form a near-planar bonded network. In the random case it is easily seen that the mean number of sides is four for the polygonal voids formed by the fibre process. The distribution of polygonal void sizes is then given by the direct product of independent identical exponential distributions—reflecting the Poisson processes for fibre intersections. For isotropic but *non-random* manufacturing processes, Dodson and Sampson [9] used the product of two gamma distributions to obtain the void size distribution, recovering the known random model as a special case. For some such isotropic materials it may be appropriate to consider correlated polygon sides, since the voids tend to be 'roundish' suggesting positive correlation among polygon sides. Some commercial processes involve preferential alignment of fibres in the direction of manufacture, resulting in anisotropic void distributions. In these cases the product of independent gamma distributions needs to be replaced by the McKay bivariate model and we shall report this study elsewhere.

The second application concerns tomographic images of soil samples in hydrology surveys. Such images yield 3-dimensional reconstructions of the porous structure and interestingly these generate a bivariate positively correlated pair of random variables $0 < x < y < \infty$. Here, the larger variable represents the size of a void and the smaller variable represents the sizes of the throats that connect it to neighbouring

voids. This work is being pursued in collaboration with Professor J. Scharkanski of the Federal University of Rio Grande do Sul, Brazil and will be reported elsewhere [3].

The authors used *Mathematica* to perform analytic calculations and can make available working notebooks for others to use.

8 Concluding remarks

We have formalised the concept of a process being 'nearly random' by proving that in the subspace information metric topology of Euclidean $\mathbb{R}^3$, every neighbourhood of an exponential distribution contains a neighbourhood of gamma distributions. Moreover, every neighbourhood of the uniform distribution contains a neighbourhood of log-gamma distributions.

We have derived the information geometry of the 3-manifold M of McKay bivariate gamma distributions, which can provide a metrization of departures from randomness and independence for bivariate processes. Additionally, we give the metric for a 5-manifold version that includes location parameters for the two random variables. The curvature objects are derived for M and those on three submanifolds that illustrate some cases of possible practical interest. As in the case of bivariate normal manifolds, we have negative scalar curvature on the McKay bivariate gamma manifolds, but here it is not constant and we show how it depends on correlation in two cases. These results have applications, for example, in the characterization of stochastic materials.

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