CONNECTIONS ON PRINCIPAL S^1–BUNDLES OVER COMPACTA*

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1 Introduction

Consider a principal G–bundle $G \hookrightarrow P \xrightarrow{\pi}\!\!\!\!\rightarrow B$. Principal connections are characterized in terms of jets by the following.

Theorem (García [5]) *There exists a natural one–to–one correspondence between the set of principal connections on P and the set of G–invariant sections of the first jet bundle $JP \longrightarrow\!\!\!\!\rightarrow P$; that is, sections of $JP/G \longrightarrow\!\!\!\!\rightarrow B$. Moreover, every principal connection on P appears as the pullback of a certain universal connection $\boldsymbol{\omega}_{\boldsymbol{\Lambda}}$ on $JP/G \times P \longrightarrow\!\!\!\!\rightarrow JP/G$.*

In this paper, we characterize the space of principal connections in the cases of the Heisenberg bundles over T^2 and the Hopf bundle over S^2.

For S^1–bundles over compacta, we have

Theorem (Kobayashi [7]) *Let M be a compact manifold. Then there is a one–to–one correspondence between equivalence classes of circle bundles over M and the cohomology group $H^2(M, \mathbb{Z})$. Furthermore, given an integral closed 2–form Φ on M there is a circle bundle $\pi\colon E \longrightarrow M$ with connection form $\boldsymbol{\omega}$ such that Φ is the curvature of $\boldsymbol{\omega}$ (that is $\pi^*(\Phi) = d\boldsymbol{\omega}$).*

We bring these two theorems together by displaying the universal connections and their universal curvatures, so explicitly illustrating Chern–Weil theory in our context.

In a subsequent paper, we shall apply the method introduced here to reductive coset spaces in general.

2 S^1–bundles over T^2

We define H_0^3 to be the 3–dimensional abelian Lie group, and for $n \neq 0$ we define

$$H_n^3 = \left\{ \begin{pmatrix} 1 & x^1 & -\frac{x^3}{n} \\ 0 & 1 & x^2 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

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so that H^3_n is diffeomorphic to $\mathbb{R}^3$ but has the Lie group structure induced by matrix multiplication as indicated. Observe that in each case H^3_n is a semidirect product $\mathbb{R}^2 \ltimes_{\varphi_n} \mathbb{R}$ with the twisting given by the representation

$$\varphi_n \colon \mathbb{R} \longrightarrow \mathrm{GL}_2 \;:\; t \mapsto \begin{pmatrix} 1 & -\frac{t}{n} \\ 0 & 1 \end{pmatrix}$$

for $n \neq 0$ and the trivial representation for $n = 0$.

Theorem 1 *Up to isomorphism, all principal S^1–bundles over T^2 can be obtained as*

$$K_n = H^3_n/\mathbb{Z}^3 \xrightarrow{\pi} T^2 = \mathbb{R}^2/\mathbb{Z}^2 \ .$$

Proof. Representing $S^1 = \mathbb{R}/\mathbb{Z}$, we apply reduction mod 1 to regard $\varphi_n \colon S^1 \longrightarrow$ *Diff* (T^2), the diffeomorphism group of the torus. Then from the covering action

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\varphi_n} & \mathbb{R}^2 \\ \downarrow & & \downarrow \\ T^2 & \xrightarrow{\varphi_n} & T^2 \end{array}$$

we obtain the pullback diagram

$$\begin{array}{ccc} T^2 \times_{\varphi_n} S^1 & \longrightarrow & T^2 \times S^1 \\ \downarrow & & \downarrow \\ T^2 & \xrightarrow{\varphi_n} & T^2 \end{array}$$

Observing that $T^2 \times_{\varphi_n} S^1 \cong H^3_n/\mathbb{Z}^3$, the theorem follows. □

Thus we consider the Heisenberg bundles

$$H^3_n \xrightarrow{\tilde{\pi}} \mathbb{R}^2 \;:\; (x^1, x^2, x^3) \mapsto (x^1, x^2) \ .$$

Now, the left–invariant vector fields on H^3_n are

$$e_1 = \frac{\partial}{\partial x^1} \ , \quad e_3 = \frac{\partial}{\partial x^3} \quad \text{and} \quad e_2 = \frac{\partial}{\partial x^2} - n x^1 \frac{\partial}{\partial x^3} \ .$$

We denote the dual left–invariant 1–forms by ω^1, ω^3, ω^2, respectively, so that the Maurer–Cartan equations appear as

$$\begin{array}{rcl} d\omega^1 & = & 0 \ , \\ d\omega^2 & = & 0 \ , \\ d\omega^3 & = & n\,\omega^1 \wedge \omega^2 \ . \end{array}$$

This reflects the splitting of the Heisenberg algebra into a semidirect sum

$$\mathfrak{h}^3_n = \mathbb{R}^2 \oplus_{\varphi_{n*}} \mathbb{R}$$

of abelian Lie algebras.

We induce connections on these Heisenberg bundles *via* the vertical and the horizontal bundles

$$\begin{array}{rcl} \mathcal{V}_n & = & \langle e_3 \rangle \\ \mathcal{H}_n & = & \langle e_1, e_2 \rangle \end{array}$$

so that the corresponding connection form is $\boldsymbol{\omega} = \omega^3$. Then the curvature form is $\boldsymbol{\Omega} = d\omega^3 = n\,\omega^1 \wedge \omega^2$.

Next, observe that the lift of the action of $\mathbb{R}$ on H^3_n, which is given by

$$t \cdot \begin{pmatrix} 1 & x^1 & -\frac{x^3}{n} \\ 0 & 1 & x^2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x^1 & -\frac{x^3}{n} + t \\ 0 & 1 & x^2 \\ 0 & 0 & 1 \end{pmatrix} ,$$

to TH^3_n and to the first jet bundle of $H^3_n \xrightarrow{\tilde{\pi}} \mathbb{R}^2$, denoted by $J(\tilde{\pi})$, is trivial. Thus a section of $\tilde{\pi}$ may be represented by a function $s\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$, and a section of $J(\tilde{\pi})$ by the triple (x, s, σ) where $x = (x^1, x^2)$, s is a section of $\tilde{\pi}$, and $\sigma\colon T_x\mathbb{R}^2 \longrightarrow T_{s(x)}\mathbb{R}$. With respect to the frame determined by the coordinates x^1 and x^2 on the base and the vertical vector e_3 tangent to the fiber, σ appears as a 1×2 matrix (σ_1, σ_2). In these coordinates, the horizontal lift map of the corresponding connection is represented by

$$\begin{pmatrix} 1 & 0 \\ \sigma_1 & \sigma_2 \end{pmatrix} ,$$

reflecting the privileged role of the x^1–coordinate in H^3_n.

In order to discuss S^1–bundles over T^2, we must pass to the quotient bundle $H^3_n/\mathbb{Z}^3 \xrightarrow{\pi} T^2 = \mathbb{R}^2/\mathbb{Z}^2$. This forces both σ_1 and σ_2 to be periodic of period 1, whence the two components of s_* must also be periodic of period 1. Thus s is a doubly periodic function on $\mathbb{R}^2$ and we have proved

Theorem 2 *The space of all principal connections on H^3_n over $\mathbb{R}^2$ is represented by the set of all $\mathbb{R}^2$–valued functions $\sigma = (\sigma_1, \sigma_2)$ on $\mathbb{R}^2$ such that σ_2 is nonvanishing, and the space of all principal connections on $K_n = H^3_n/\mathbb{Z}^3$ over $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ by the set of all $\mathbb{R}^2/\mathbb{Z}^2$–valued σ on $\mathbb{R}^2/\mathbb{Z}^2$ such that σ_2 is nonvanishing.*

Since we view principal connections as horizontal lift maps, the connection 1–form is obtained from

$$\mathbf{1} - \begin{pmatrix} 1 & 0 \\ \sigma_1 & \sigma_2 \end{pmatrix}$$

by a brief calculation as

$$\boldsymbol{\omega} = \omega^3 - \sigma_1\,\omega^1 + (1 - \sigma_2)\,\omega^2 \ . \tag{1}$$

The special choices $\sigma_1 = 0$, $\sigma_2 = 1$ yield the canonical connections on these Heisenberg bundles.

Computing the curvature, we obtain

$$\boldsymbol{\Omega}_{\boldsymbol{\omega}} = d\boldsymbol{\omega} = n\,\omega^1 \wedge \omega^2 - d(\sigma_1\,\omega^1 + \sigma_2\,\omega^2) \ . \tag{2}$$

As expected from the Chern–Weil theory, the Euler class does not depend on the connection. Observe that in fact we have obtained a stronger, more explicit result: we see directly that the curvature of any connection is cohomologous to that of the canonical connection.

Observe that (

Modugno [8] extended García's theorem to general fibered manifolds. He identified the spaces of sections which characterize connections and incorporated them into *systems* of connections, on which he found universal connections. Applications to the particular case of the frame bundle have yielded some stability theorems [1, 4].

Remark Circle bundles over other surfaces of genus ≥ 1 may be handled in a similar manner.

3 The Hopf Fibration

Consider these equivalent versions of the famous Hopf bundle:

$$\begin{array}{ccccccc} S^3 & \cong & \mathrm{SO}_4/\mathrm{SO}_3 & \cong & (\mathrm{SO}_3 \times S^3)/\mathrm{SO}_3 & \cong & (\mathrm{SO}_3 \times \mathrm{Spin}_3)/\mathrm{SO}_3 \\ \pi\downarrow & & \pi\downarrow & & \pi\downarrow & & \pi\downarrow \\ S^2 & \cong & \mathrm{SO}_3/\mathrm{SO}_2 & \cong & \mathrm{SO}_3/\mathrm{SO}_2 & \cong & \mathrm{SO}_3/\mathrm{SO}_2 \end{array}$$

First we work with the bundle $\mathrm{SO}_4 = \mathrm{SO}_3 \times \mathrm{Spin}_3 \xrightarrow{\tilde{\pi}} \mathrm{SO}_3$.

We have the Lie algebra splitting

$$\mathrm{so}_4 \cong \mathrm{so}_3 \oplus so_3$$

with Maurer–Cartan equations

$$\left\{ \begin{array}{ccc} d\omega^1 & = & \omega^2 \wedge \omega^3 \\ d\omega^2 & = & \omega^3 \wedge \omega^1 \\ d\omega^3 & = & \omega^1 \wedge \omega^2 \end{array} \right\} \bigoplus \left\{ \begin{array}{ccc} d\omega^4 & = & \omega^5 \wedge \omega^6 \\ d\omega^5 & = & \omega^6 \wedge \omega^4 \\ d\omega^6 & = & \omega^4 \wedge \omega^5 \end{array} \right\}$$

where the (ω^i) are a basis for the left–invariant 1–forms on so_4 and we have chosen all 6 structure constants to be 1/2.

Define a basis for left–invariant 1–forms on SO_3, α^1, α^2, α^3, such that for $\tilde{\pi}\colon \mathrm{SO}_4 \twoheadrightarrow \mathrm{SO}_3$, we have $\tilde{\pi}^*\alpha^i = \omega^{i+3}$.

Next we take the quotients to obtain $\pi\colon \mathrm{SO}_4/\mathrm{SO}_3 \twoheadrightarrow \mathrm{SO}_3/\mathrm{SO}_2$.

For convenience, we consider the subgroup $\mathrm{SO}_2 \prec \mathrm{SO}_3$ *via* its corresponding Lie subalgebra

$$\mathrm{so}_2 = \left\langle \frac{1}{3}\left(\alpha^1 + \alpha^2 + \alpha^3\right) \right\rangle \ ;$$

that is,

$$\mathrm{SO}_2 = \left\{ \exp\left(\frac{t}{3}\left(\varepsilon^1 + \varepsilon^2 + \varepsilon^3\right) \right) \right\}$$

so

$$\pi^*\mathrm{so}_2 = \langle \omega^4 + \omega^5 + \omega^6 \rangle \ .$$

This will give rise to the Hopf fibration. To obtain the remainder of the principal S^1–bundles over S^2, we simply require

$$\pi^*\left(\frac{1}{3}(\alpha^1 + \alpha^2 + \alpha^3) \right) = n\left(\omega^4 + \omega^5 + \omega^6\right)$$

for integral $n \neq 0$, and take instead the trivial bundle $S^2 \times S^1 \twoheadrightarrow S^2$ for $n = 0$. The Hopf bundle is then the case $n = 1$.

3.1 The Hopf Connection

Observe that we have a simple connection form $\tilde{\boldsymbol{\omega}}_H$ on $\mathrm{SO}_4 \xrightarrow{\tilde{\pi}} \mathrm{SO}_3$ as follows. Dual to the coframe (ω^i) we take the frame (e_i), obtaining

$$\mathcal{V}_{\tilde{\boldsymbol{\omega}}_H} \oplus \mathcal{H}_{\tilde{\boldsymbol{\omega}}_H} = T(\mathrm{SO}_4)$$

with

$$\mathcal{V}_{\tilde{\boldsymbol{\omega}}_H} = \langle e_1, e_2, e_3 \rangle \quad , \quad \mathcal{H}_{\tilde{\boldsymbol{\omega}}_H} = \langle e_4, e_5, e_6 \rangle \ .$$

Note that our frames are *anholonomic*; they are adapted to the group structure, not to the connection.

Now, to obtain a connection on $\mathrm{SO}_4 \longrightarrow \mathrm{SO}_3/\mathrm{SO}_2$, we simply "move" a 1–dimension subspace from $\mathcal{H}_{\tilde{\boldsymbol{\omega}}_H}$ to become vertical. Changing from total space SO_4 to $\mathrm{SO}_4/\mathrm{SO}_3$ only involves removing $\langle e_1, e_2, e_3 \rangle$ from the resulting vertical space. Our representative of so_2 is $\langle \frac{1}{3}\left(\alpha^1 + \alpha^2 + \alpha^3\right) \rangle$, so we obtain a connection

$$\boldsymbol{\omega}_H = \omega^4 + \omega^5 + \omega^6$$

on $\mathrm{SO}_4/\mathrm{SO}_3 \longrightarrow \mathrm{SO}_3/\mathrm{SO}_2$ with

$$\mathcal{V}_{\boldsymbol{\omega}_H} = \langle e_4 + e_5 + e_6 \rangle$$

and $\mathcal{H}_{\boldsymbol{\omega}_H}$ a 2–dimensional subbundle of $\langle e_4, e_5, e_6 \rangle$ with "normal vector" in the direction $(1, 1, 1)$. Its curvature form is

$$\boldsymbol{\Omega}_H = d\boldsymbol{\omega}_H = \omega^4 \wedge \omega^5 + \omega^5 \wedge \omega^6 + \omega^6 \wedge \omega^4 \ .$$

The corresponding (dual) horizontal lift map on covectors is

$$\boldsymbol{\omega}_H^{\uparrow} \colon \alpha^i + \left\langle \frac{1}{3}\left(\alpha^1 + \alpha^2 + \alpha^3\right) \right\rangle \longmapsto \omega^{i+3} + \langle \omega^4 + \omega^5 + \omega^6 \rangle$$

which is characterized by the 3×3 identity matrix, so distinguishing the *Hopf connection* $\boldsymbol{\omega}_H$.

For the remainder of the principal S^1–bundles over S^2, we take

$$\boldsymbol{\omega} = n\,\boldsymbol{\omega}_H = n\left(\omega^4 + \omega^5 + \omega^6\right)$$

for the connection 1–form, obtaining $\boldsymbol{\Omega} = n\,\boldsymbol{\Omega}_H$ for the curvature 2–form. We observe that the bundle corresponding to n then has Euler class n, as it should.

3.2 The Space of Connections

The space of all principal connections on the Hopf bundle is approached *via* that for the $\mathrm{SO}_3 \times \mathrm{Spin}_3 \cong \mathrm{SO}_4 \xrightarrow{\tilde{\pi}} \mathrm{SO}_3$ bundle. A connection corresponds to a section of the first jet bundle, and such a section associates to each $x \in \mathrm{SO}_3$ a point $s(x) \in \mathrm{Spin}_3$ and a map $T_x\mathrm{SO}_3 \longrightarrow T_{s(x)}\mathrm{Spin}_3$. This latter, as a matrix with respect to the domain frame (ε_i) dual to the coframe (α^i) and the codomain frame (e_4, e_5, e_6), we denote by (σ_{ij}).

To pass to the Hopf bundle, $s \colon \mathrm{SO}_3 \longrightarrow \mathrm{Spin}_3$ must be SO_2–equivariant; this is given by the infinitesimal condition

$$s^*(\omega^4 + \omega^5 + \omega^6) = \alpha^1 + \alpha^2 + \alpha^3 \ .$$

It follows that, as a matrix, (s^*) must have column–sums equal to 1 and its transpose (s_*) has row–sums equal to 1, whence (σ_{ij}) must have row–sums equal to 1 for passage to the Hopf bundle. Hence, these sections of the first jet bundle of $SO_3 \times Spin_3 \xrightarrow{\tilde{\pi}} SO_3$ which have such s and (σ_{ij}) correspond to SO_2–equivariant sections of the first jet bundle of $SO_4/SO_3 \xrightarrow{\pi} SO_3/SO_2$, and these are principal connections. This proves

Theorem 3 *The space of all principal connections on the Hopf bundle is represented by the space of all* 3×3 *matrices* (σ_{ij}) *of rank* 3 *such that the row–sums are* 1.

Again, we obtain the connection 1–forms from $(\mathbf{1} - \boldsymbol{\omega}^{\uparrow})$. This is essentially given by $\mathbf{1} - (\sigma_{ij})$; we must pass to the quotient, of course, for the actual Hopf bundle. For the bundle $SO_4 \xrightarrow{\tilde{\pi}} SO_3$, we have

$$\begin{aligned}\tilde{\boldsymbol{\omega}} &= \tilde{\boldsymbol{\omega}}_H + (1 - (\sigma_{ij})) \cdot \begin{pmatrix} \omega^4 \\ \omega^5 \\ \omega^6 \end{pmatrix} \\ &= \tilde{\boldsymbol{\omega}}_H + \sum_{i,j} (\delta_{ij} - \sigma_{ij})\, \omega^{j+3} \ , \\ \tilde{\boldsymbol{\Omega}} &= d\tilde{\boldsymbol{\omega}} \ = \ d\tilde{\boldsymbol{\omega}}_H + d\left(\sum_{i,j} (\delta_{ij} - \sigma_{ij})\, \omega^{j+3} \right) \ ,\end{aligned}$$

again displaying the curvature of any connection as cohomologous to that of a standard connection.

For the Hopf bundle itself, by a suitable change of frames we may assume that $(\sigma_{ij}) = (\sigma_{11}) \oplus (\sigma'_{ij})$ in block–diagonal form, where $\sigma_{11} \neq 0$ and $(\sigma'_{ij}) \in GL_2$. Then (σ'_{ij}) gives the horizontal lift map $\boldsymbol{\omega}_H^{\uparrow}$ for the Hopf connection, and we obtain any other connection on the Hopf bundle as

$$\begin{aligned}\boldsymbol{\omega} &= \boldsymbol{\omega}_H + (\mathbf{1} - (\sigma'_{ij})) \cdot \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} \ , \\ \boldsymbol{\Omega} &= d\boldsymbol{\omega} \ = \ \boldsymbol{\Omega}_H + d\left(\sum_{i,j} (\delta_{ij} - \sigma'_{ij}) \eta^j \right)\end{aligned}$$

where η^1, η^2, $\omega^4 + \omega^5 + \omega^6$ are the new basis of left–invariant 1–forms on $Spin_3 = S^3$. Once again, the curvature of any connection is explicitly displayed as cohomologous to that of a standard connection, the Hopf connection here.

Finally, any connection on any of the other principal S^1–bundles over S^2 is obtained merely by substituting $n\,\boldsymbol{\omega}_H$ for $\boldsymbol{\omega}_H$, and thus $n\,\boldsymbol{\Omega}_H$ for $\boldsymbol{\Omega}_H$, reflecting our standard connection $n\,\boldsymbol{\omega}_H$ on the bundle corresponding to n.

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