Alpha connections and an affine embedding of the McKay bivariate gamma 3-manifold

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Abstract The McKay bivariate gamma distribution has marginal gamma densities with positive covariance and recently its information geometry as a 3-manifold has been provided. Here we derive: natural coordinates, explicit expressions for the α-connections, mutually dual foliations and an affine embedding in Euclidean $\mathbb{R}^4$. We compute also the Kullback-Leibler divergence and compare it with the canonical divergence for two McKay densities.

1 Introduction

Recently, Dodson and Matsuzoe [7] provided an affine embedding for gamma densities and Arwini and Dodson [3] derived the information geometry of the McKay family of bivariate gamma densities. We showed in [3] that gamma distributions provide models for departures from randomness since every neighbourhood of an exponential distribution contains a neighbourhood of gamma distributions, using the information theoretic metric topology. This topological result is inherently stable under small perturbations of a process, something that would be important in real applications. It gives confidence in the use of gamma distributions to model near random processes. We recall also the result of Hwang and Hu [8] who proved for $n \geq 3$ independent positive random variables $x_1, x_2, \ldots, x_n$ with a common continuous probability density function f, that *having independence of the sample mean $\bar{x}$ and sample coefficient of variation $cv = S/\bar{x}$ is equivalent to f being a gamma distribution.* The gamma manifold geometry has been applied to a range of

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real situations, such as genome structure [4], cosmological voids and galaxy clustering [5] and communication clustering[6]. For further details on information geometry we refer to Amari [1] and Amari and Nagaoka [2].

The set of McKay bivariate gamma distributions is

$$\begin{aligned} M \;&=\; \{f \mid f(x,y;\alpha_1,\sigma_{12},\alpha_2) = \left(\frac{\alpha_1}{\sigma_{12}}\right)^{\frac{\alpha_1+\alpha_2}{2}} \frac{x^{\alpha_1-1}\,(y-x)^{\alpha_2-1}}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)}\, e^{-\sqrt{\frac{\alpha_1}{\sigma_{12}}}y}\,, \\ &\qquad y>x>0,\ \alpha_1,\sigma_{12},\alpha_2>0\}. \end{aligned} \tag{1.1}$$

Take $(\alpha_1,\sigma_{12},\alpha_2) = (\xi^1,\xi^2,\xi^3) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ as a local coordinate system for (1.1), with σ_{12} the covariance. Then M is a Riemannian 3-manifold with Fisher metric given by [3]:

$$[g_{ij}] = \left[\int_0^\infty \int_x^\infty \left(\frac{\partial^2 \log f}{\partial \xi^i \partial \xi^j}\right) f\ dy\ dx\right] = \begin{bmatrix} \frac{-3\,\alpha_1+\alpha_2}{4\,{\alpha_1}^2} + \phi'(\alpha_1) & \frac{\alpha_1-\alpha_2}{4\,\alpha_1\,\sigma_{12}} & -\frac{1}{2\,\alpha_1} \\ \frac{\alpha_1-\alpha_2}{4\,\alpha_1\,\sigma_{12}} & \frac{\alpha_1+\alpha_2}{4\,{\sigma_{12}}^2} & \frac{1}{2\,\sigma_{12}} \\ -\frac{1}{2\,\alpha_1} & \frac{1}{2\,\sigma_{12}} & \phi'(\alpha_2) \end{bmatrix} \tag{1.2}$$

where $\phi(\alpha_i) = \frac{\Gamma'(\alpha_i)}{\Gamma(\alpha_i)} \quad (i=1,2)$.

The information geometry of M can provide a metrization of departures from randomness and departures from independence for bivariate processes with positive covariance. They have applications, for example, in the characterization of stochastic materials.

2 Natural coordinate systems, potential functions and α-connections

For each $\alpha \in \mathbb{R}$, the α *(or $\nabla^{(\alpha)}$)-connection* is the torsion-free affine connection with components:

$$\Gamma^{(\alpha)}_{ij,k} = \int_0^\infty \int_x^\infty \left(\frac{\partial^2 \log f}{\partial \xi^i \partial \xi^j}\,\frac{\partial \log f}{\partial \xi^k} + \frac{1-\alpha}{2}\,\frac{\partial \log f}{\partial \xi^i}\,\frac{\partial \log f}{\partial \xi^j}\,\frac{\partial \log f}{\partial \xi^k}\right) f\ dy\ dx$$

In particular, the 1-connection is said to be an *exponential connection*, and the (-1)-connection is said to be a *mixture connection*. We say that an α-connection and the $(-\alpha)$-connection are *mutually dual* with respect to the Fisher metric g since the following formula holds:

$$Xg(Y,Z) = g(\nabla^{(\alpha)}_X Y, Z) + g(Y, \nabla^{(-\alpha)}_X Z),$$

where X, Y and Z are arbitrary vector fields on M.

Proposition 2.1 *Let M be the McKay bivariate gamma 3-manifold. Set $\theta = \sqrt{\frac{\alpha_1}{\sigma_{12}}}$. Then $(\alpha_1,\theta,\alpha_2)$ is a natural coordinate system of the 1-connection and*

$$\psi = \log\Gamma(\alpha_1) + \log\Gamma(\alpha_2) - (\alpha_1+\alpha_2)\,\log\theta \tag{2.3}$$

is the corresponding potential function.

Proof: Set $\theta = \sqrt{\frac{\alpha_1}{\sigma_{12}}}$. Then the logarithm of McKay bivariate gamma distributions can be written as

$$\begin{aligned}
\log f(x, y; \alpha_1, \theta, \alpha_2) &= \log\left(\frac{\theta^{(\alpha_1+\alpha_2)} x^{\alpha_1-1} (y-x)^{\alpha_2-1} e^{-\theta y}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\right) \\
&= -\log x - \log(y-x) \qquad (2.4) \\
&\quad +\alpha_1(\log x) + \theta(-y) + \alpha_2(\log(y-x)) \qquad (2.5) \\
&\quad -(\log\Gamma(\alpha_1) + \log\Gamma(\alpha_2) - (\alpha_1+\alpha_2)\log\theta). \qquad (2.6)
\end{aligned}$$

Hence the set of all McKay bivariate gamma distributions is an exponential family. The terms in the line (2.5) implies that $(\alpha_1, \theta, \alpha_2)$ is a natural coordinate system, and (2.6) implies that $\psi = \log\Gamma(\alpha_1) + \log\Gamma(\alpha_2) - (\alpha_1+\alpha_2)\log\theta$ is its potential function. □

We remark that (2.4) implies that $C = -\log x - \log(y-x)$ is the normalization function since f is a probability distribution.

Proposition 2.2 *The Fisher metric with respect to natural coordinates* $(\alpha_1, \theta, \alpha_2)$ *is given by:*

$$[g_{ij}] = \begin{bmatrix} \phi'(\alpha_1) & -\frac{1}{\theta} & 0 \\ -\frac{1}{\theta} & \frac{\alpha_1+\alpha_2}{\theta^2} & -\frac{1}{\theta} \\ 0 & -\frac{1}{\theta} & \phi'(\alpha_2) \end{bmatrix} \qquad (2.7)$$

Proof: Since ψ is a potential function, the Fisher metric is given by the Hessian of ψ, that is,

$$g_{ij} = \frac{\partial^2\psi}{\partial\xi^i\partial\xi^j}.$$

Then, we have the Fisher metric by a straightforward calculation. □

Since the set of Mckay bivariate gamma distributions is an exponential family, the connection $\nabla^{(1)}$ is flat. In this case, $(\alpha_1, \theta, \alpha_2)$ is a 1-affine coordinate system.

Proposition 2.3 *With respect to natural coordinates* $(\alpha_1, \theta, \alpha_2)$, *the components of the* α*-connection and its curvature tensor on the Mckay bivariate gamma manifold are as* *follows:*

$$
\begin{aligned}
\Gamma^{(\alpha)}_{11,1} &= \frac{(1-\alpha)\ \phi''(\alpha_1)}{2}\,, \\
\Gamma^{(\alpha)}_{22,1} &= \Gamma^{(\alpha)}_{12,2} = \frac{(1-\alpha)}{2\,\theta^2}\,, \\
\Gamma^{(\alpha)}_{22,2} &= -\frac{(1-\alpha)\ (\alpha_1+\alpha_2)}{\theta^3}, \\
\Gamma^{(\alpha)}_{23,2} &= \Gamma^{(\alpha)}_{22,3} = \Gamma^{(\alpha)}_{22,1}\,, \\
\Gamma^{(\alpha)}_{33,3} &= \frac{(1-\alpha)\ \phi''(\alpha_2)}{2}\,, \\
\Gamma^{(\alpha)1}_{11} &= \frac{(-1+\alpha)\ \phi''(\alpha_1)\ (-1+\phi'(\alpha_2)\ (\alpha_1+\alpha_2))}{2\ (\phi'(\alpha_1)+\phi'(\alpha_2)-\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))}\,, \\
\Gamma^{(\alpha)2}_{11} &= \frac{\theta\ (-1+\alpha)\ \phi'(\alpha_2)\,\phi''(\alpha_1)}{2\ (\phi'(\alpha_1)+\phi'(\alpha_2)-\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))}\,, \\
\Gamma^{(\alpha)3}_{11} &= \frac{(-1+\alpha)\ \phi''(\alpha_1)}{2\ (\phi'(\alpha_1)+\phi'(\alpha_2)-\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))}\,, \\
\Gamma^{(\alpha)1}_{12} &= \Gamma^{(\alpha)1}_{23} = \frac{(-1+\alpha)\ \phi'(\alpha_2)}{2\,\theta\ (\phi'(\alpha_1)+\phi'(\alpha_2)-\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))}\,, \\
\Gamma^{(\alpha)2}_{12} &= \Gamma^{(\alpha)2}_{23} = \frac{(-1+\alpha)\ \phi'(\alpha_1)\,\phi'(\alpha_2)}{2\ (\phi'(\alpha_1)+\phi'(\alpha_2)-\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))}\,, \\
\Gamma^{(\alpha)3}_{12} &= \Gamma^{(\alpha)3}_{23} = \frac{(-1+\alpha)\ \phi'(\alpha_1)}{2\,\theta\ (\phi'(\alpha_1)+\phi'(\alpha_2)-\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))}\,, \\
\Gamma^{(\alpha)1}_{22} &= \frac{-\left((-1+\alpha)\ \phi'(\alpha_2)\ (\alpha_1+\alpha_2)\right)}{2\,\theta^2\ (\phi'(\alpha_1)+\phi'(\alpha_2)-\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))}\,, \\
\Gamma^{(\alpha)2}_{22} &= \frac{-\left((-1+\alpha)\ (-\phi'(\alpha_1)-\phi'(\alpha_2)+2\,\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))\right)}{2\,\theta\ (\phi'(\alpha_1)+\phi'(\alpha_2)-\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))}\,, \\
\Gamma^{(\alpha)3}_{22} &= \frac{-\left((-1+\alpha)\ \phi'(\alpha_1)\ (\alpha_1+\alpha_2)\right)}{2\,\theta^2\ (\phi'(\alpha_1)+\phi'(\alpha_2)-\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))}\,, \\
\Gamma^{(\alpha)1}_{33} &= \frac{(-1+\alpha)\ \phi''(\alpha_2)}{2\ (\phi'(\alpha_1)+\phi'(\alpha_2)-\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))}\,, \\
\Gamma^{(\alpha)2}_{33} &= \frac{\theta\ (-1+\alpha)\ \phi'(\alpha_1)\,\phi''(\alpha_2)}{2\ (\phi'(\alpha_1)+\phi'(\alpha_2)-\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))}\,, \\
\Gamma^{(\alpha)3}_{33} &= \frac{(-1+\alpha)\ \phi''(\alpha_2)\ (-1+\phi'(\alpha_1)\ (\alpha_1+\alpha_2))}{2\ (\phi'(\alpha_1)+\phi'(\alpha_2)-\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))}\,, \\
R_{1212} &= \frac{(1-\alpha^2)\ \phi'(\alpha_2)\ (\phi'(\alpha_1)+\phi''(\alpha_1)\ (\alpha_1+\alpha_2))}{4\,\theta^2\ (\phi'(\alpha_1)+\phi'(\alpha_2)-\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))}\,, \\
R_{1213} &= -\frac{(1-\alpha^2)\ \phi'(\alpha_2)\,\phi''(\alpha_1)}{4\,\theta\ (\phi'(\alpha_1)+\phi'(\alpha_2)-\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))}\,, \\
R_{1223} &= -\frac{(1-\alpha^2)\ \phi'(\alpha_1)\,\phi'(\alpha_2)}{4\,\theta^2\ (\phi'(\alpha_1)+\phi'(\alpha_2)-\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))}\,, \\
R_{1313} &= -\frac{(1-\alpha^2)\ \phi''(\alpha_1)\,\phi''(\alpha_2)}{4\ (\phi'(\alpha_1)+\phi'(\alpha_2)-\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))}\,, \\
R_{1323} &= -\frac{(1-\alpha^2)\ \phi'(\alpha_1)\,\phi''(\alpha_2)}{4\,\theta\ (\phi'(\alpha_1)+\phi'(\alpha_2)-\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))}\,, \\
R_{2323} &= \frac{(1-\alpha^2)\ \phi'(\alpha_1)\ (\phi'(\alpha_2)+\phi''(\alpha_2)\ (\alpha_1+\alpha_2))}{4\,\theta^2\ (\phi'(\alpha_1)+\phi'(\alpha_2)-\phi'(\alpha_1)\,\phi'(\alpha_2)\ (\alpha_1+\alpha_2))} \qquad (2.8)
\end{aligned}
$$

while the other independent components are zero. □

So the 1 and (-1)-connections on the Mckay manifold are flat.

3 Mutually dual foliations

We give a mutually dual foliation of the McKay bivariate gamma manifold. Since M is an exponential family, a mixture coordinate system is given by the potential function (2.3), that is,

$$\begin{aligned} \eta_1 &= \frac{\partial \psi}{\partial \alpha_1} = \phi(\alpha_1) - \log\theta, \\ \eta_2 &= \frac{\partial \psi}{\partial \theta} = -\frac{\alpha_1+\alpha_2}{\theta}, \\ \eta_3 &= \frac{\partial \psi}{\partial \alpha_2} = \phi(\alpha_2) - \log\theta. \end{aligned} \tag{3.9}$$

Since $(\alpha_1, \theta, \alpha_2)$ is a 1-affine coordinate system, (η_1, η_2, η_3) is a (-1)-affine coordinate system, and they are mutually dual with respect to the Fisher metric. Therefore the McKay bivariate gamma manifold has dually orthogonal foliations (See Section 3.7 in [2]).

The coordinates in (3.9) have a potential function given by:

$$\lambda = \alpha_1\phi(\alpha_1) + \alpha_2\phi(\alpha_2) - (\alpha_1+\alpha_2) - \log\Gamma(\alpha_1) - \log\Gamma(\alpha_2). \tag{3.10}$$

Example 3.1 *Take $(\alpha_1, \eta_2, \alpha_2)$ as a coordinate system for M; the McKay bivariate gamma distibutions take the form:*

$$f(x,y;\alpha_1,\eta_2,\alpha_2) = \left(-\frac{\alpha_1+\alpha_2}{\eta_2}\right)^{\alpha_1+\alpha_2} \frac{x^{\alpha_1-1}\,(y-x)^{\alpha_2-1}}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)}\, e^{\frac{\alpha_1+\alpha_2}{\eta_2}y}.$$

and the Fisher metric is

$$\left[\begin{array}{ccc} \phi'(\alpha_1) - \frac{1}{\alpha_1+\alpha_2} & 0 & -\frac{1}{\alpha_1+\alpha_2} \\ 0 & \frac{\alpha_1+\alpha_2}{(\eta_2)^2} & 0 \\ -\frac{1}{\alpha_1+\alpha_2} & 0 & \phi'(\alpha_2) - \frac{1}{\alpha_1+\alpha_2} \end{array}\right].$$

We remark that $(\alpha_1, \theta, \alpha_2)$ is a geodesic coordinate system of $\nabla^{(1)}$, and (η_1, η_2, η_3) is a geodesic coordinate system of $\nabla^{(-1)}$.

4 Affine immersions

We show that the McKay bivariate gamma manifold can be realized in Euclidean $\mathbb{R}^4$ by an affine immersion.

Let M be an n-dimensional manifold. Let f be an immersion from M to $\mathbb{R}^{n+1}$, and let ξ be a local vector field along f. If ξ is transversal to $f(M)$ everywhere, we call the pair $\{f, \xi\}$ an *affine immersion.* We also call ξ a *transversal vector field.*

Denote by D the standard flat affine connection of $\mathbb{R}^{n+1}$. By the decomposition of the tangent space $T_{f(x)}\mathbb{R}^{n+1}$ ($x \in M$), the covariant derivatives appear as follows:

$$\begin{aligned} D_X f_* Y &= f_*(\nabla_X Y) + g(X,Y)\xi, \\ D_X \xi &= -f_*(SX) + \mu(X)\xi. \end{aligned}$$

Then, an affine immersion induces a torsion free affine connection ∇, a symmetric $(0,2)$-tensor field g, a $(1,1)$-tensor field S and a 1-form μ on the given manifold M.

Proposition 4.1 *Let M be the McKay bivariate gamma manifold with the Fisher metric g and the exponential connection $\nabla^{(1)}$. Denote by $(\alpha_1, \theta, \alpha_2)$ a natural coordinate system. Then M can be realized in $\mathbb{R}^4$ by the graph of a potential function, namely, M can be realized by the affine immersion $\{f, \xi\}$:*

$$f : M \to \mathbb{R}^4 : \begin{bmatrix} \alpha_1 \\ \theta \\ \alpha_2 \end{bmatrix} \mapsto \begin{bmatrix} \alpha_1 \\ \theta \\ \alpha_2 \\ \psi \end{bmatrix}, \quad \xi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

where ψ is the potential function $\psi = \log\Gamma(\alpha_1) + \log\Gamma(\alpha_2) - (\alpha_1 + \alpha_2)\log\theta$.

Proof: The connections $\nabla^{(1)}$ and $\nabla^{(-1)}$ are torsion-free, and mutually dual with respect to g. We have the Codazzi equation:

$$(\nabla^{(1)}_X g)(Y,Z) = (\nabla^{(1)}_Y g)(X,Z). \tag{4.11}$$

Since $\nabla^{(1)}$ is a flat affine connection, the integrability condition of the affine immersion (Chapter 2 in [12]) is the Hessian symmetry

$$\frac{\partial^2 \psi}{\partial\theta^i \partial\theta^j} = \frac{\partial^2 \psi}{\partial\theta^j \partial\theta^i}, \qquad (\theta^1, \theta^2, \theta^3) = (\alpha_1, \theta, \alpha_2).$$

This partial differential equation can be solved since Equation (4.11) holds. □

5 Distance and divergences in M

The Riemannian metric defines a distance as the infimum over arc lengths in M between f_1 and f_2, and for close enough points the minimizing curve is realizable as a geodesic. In statistics, several other methods are available to represent differences between distributions in a given family; these involve so-called divergence functions.

Let $(M, g, \nabla^{(1)}, \nabla^{(-1)})$ be a dually flat space. Denote by (θ^i) the natural coordinate system (1-affine coordinate system) and by (η_i) its dual affine coordinate system. For two points f_1 and f_2 in M, the function

$$D(f_1 \parallel f_2) \;=\; \psi(f_1) + \lambda(f_2) - \sum_{i=1}^{3} \theta^i(f_1)\, \eta_i(f_2) \tag{5.12}$$

is called the *canonical divergence* or *geometric divergence*. See [2], [9], [10] and [11] for further discussion.

On the other hand,

$$KL(f_1 \parallel f_2) = \int_{y=0}^{\infty} \int_{x=0}^{y} f_1(x,y) \log \frac{f_1(x,y)}{f_2(x,y)} \, dx \, dy \tag{5.13}$$

is called the *Kullback-Leibler divergence* or *relative entropy*. We note that neither of these divergence functions is symmetric, so they do not define a distance.

However, for two points $f_1(x,y) = f_1(x,y;\alpha_1,\theta,\alpha_2)$ and $f_2(x,y) = f_2(x,y;\gamma_1,\xi,\gamma_2)$ we do have the following relation between the divergences—note the switch in order of the arguments:

$$\begin{aligned} D(f_1 \parallel f_2) &= KL(f_2 \parallel f_1) \\ &= \alpha_1 \, (\log(\xi) - \log(\theta) - \phi(\gamma_1)) + \alpha_2 \, (\log(\xi) - \log(\theta) - \phi(\gamma_2)) \\ &+ \gamma_1 \left(\frac{\theta}{\xi} + \phi(\gamma_1) - 1 \right) + \gamma_2 \left(\frac{\theta}{\xi} + \phi(\gamma_2) - 1 \right) + \log \left(\frac{\Gamma(\alpha_1)\,\Gamma(\alpha_2)}{\Gamma(\gamma_1)\,\Gamma(\gamma_2)} \right) \end{aligned} \tag{5.14}$$

where $\phi(\alpha_i) = \frac{\Gamma'(\alpha_i)}{\Gamma(\alpha_i)}$, $\phi(\gamma_i) = \frac{\Gamma'(\gamma_i)}{\Gamma(\alpha_i)}$ $\quad (i = 1, 2)$.

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