AN AFFINE EMBEDDING OF THE GAMMA MANIFOLD

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Abstract. For the space of gamma distributions with Fisher metric and exponential connections, natural coordinate systems, potential functions and an affine immersion in $\mathbb{R}^3$ are provided.

1. Introduction

Gamma manifolds provide Riemannian manifold representations of the space of gamma probability density functions. Recent applications have used such geometrical representations for models of clustering for galaxies [Do-1] and communications [Do-2], and have relevance to the evaluation of encryption devices [Do-3, Do-4]. An important feature of gamma manifolds is the geometrization of departures from randomness, because the exponential distributions, which represents Poisson processes, form a 1-dimensional submanifold. Some features of the curvature properties of gamma manifolds were known and numerical computations allowed graphical representations of geodesic sprays. In the present paper, we summarise natural coordinate systems, potential functions, and affine immersions of gamma manifolds.

2. Natural coordinate systems and potential functions

Let M be the set of gamma distributions, that is,

$$M = \left\{ p(t;\tau,\nu) \;\middle|\; p(t;\tau,\nu) = \left(\frac{\nu}{\tau}\right)^{\nu} \frac{t^{\nu-1}}{\Gamma(\nu)} e^{-t\nu/\tau},\ \tau,\nu \in \mathbb{R}^+ \right\}.$$

Identifying (τ,ν) as a local coordinate system, M can be regarded as a manifold, the *gamma manifold*. We use $(\xi^1,\xi^2) \in \mathbb{R}^+ \times \mathbb{R}^+$ to denote coordinates.

Set $l(t;\tau,\nu) = \log p(t;\tau,\nu)$; Then M, admits a Riemannian metric, the Fisher metric g, with coordinate functions:

$$g_{ij} = \int_0^\infty \frac{\partial l}{\partial \xi^i}\frac{\partial l}{\partial \xi^j} p(t;\tau,\nu)\, dt.$$

For each $\alpha \in \mathbb{R}$, we have a torsion-free affine connection with components:

$$\Gamma^{(\alpha)}_{ij,k} = \int_0^\infty \left(\frac{\partial^2 l}{\partial \xi^i \partial \xi^j}\frac{\partial l}{\partial \xi^k} - \frac{1-\alpha}{2}\frac{\partial l}{\partial \xi^i}\frac{\partial l}{\partial \xi^j}\frac{\partial l}{\partial \xi^k} \right) p(t;\tau,\nu)\, dt,$$

where $\Gamma^{(\alpha)}_{ij,k} = g(\sum_{l=1}^{2} \Gamma^{(\alpha)l}_{ij} \frac{\partial}{\partial \xi^l}, \frac{\partial}{\partial \xi^k})$. We call the affine connection an α (or a $\nabla^{(\alpha)}$)-*connection*. In particular, the 1-connection is said to be an *exponential connection*,

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and the (-1)-connection is said to be a *mixture connection*. We say that an α-connection and the $(-\alpha)$-connection are mutually *dual* with respect to the Fisher metric g (see [AN]).

Proposition 2.1 (cf. [La]). *The Fisher metric and the connection coefficients of theb α-connection on the gamma manifold with respect to (τ, ν) coordinate are given as follows:*

$$\begin{aligned}
g_{\tau\tau} &= \frac{\mu}{\tau^2}, \quad g_{\nu\nu} = \phi'(\nu) - \frac{1}{\nu}, \\
\Gamma^{(\alpha)\tau}_{\tau\tau} &= -\frac{(1+\alpha)}{\tau}, \quad \Gamma^{(\alpha)\nu}_{\tau\tau} = \frac{\alpha-1}{2\tau^2}\left(\frac{1}{-\frac{1}{\nu}+\phi'(\nu)}\right), \\
\Gamma^{(\alpha)\tau}_{\nu\tau} &= \Gamma^{(\alpha)\tau}_{\tau\nu} = \frac{\alpha+1}{2\nu}, \\
\Gamma^{(\alpha)\nu}_{\nu\nu} &= \frac{1-\alpha}{2}\left(\frac{1/\nu^2+\phi''(\nu)}{-1/\nu+\phi'(\nu)}\right),
\end{aligned}$$

where ϕ is the logarithmic derivative of the gamma function, i.e., $\phi = \Gamma'(\nu)/\Gamma(\nu)$, and all the other Christoffel symbols vanish.

Moreover, the curvature tensor of the α-connection is given as

$$R_{\tau\mu\tau\mu} = \frac{(\alpha^2-1)\{\phi(\nu)+\nu\phi(\nu)\}}{4\tau^2\nu\phi(\nu)}.$$

So the 1 and (-1)-connections on the gamma manifold are flat.

We say that a coordinate system (θ^1, θ^2) is an *affine coordinate system* if all the connection coefficients vanish and, in particular, an affine coordinate system which represents the probability density as

$$p(x;\theta^1,\theta^2) = \exp\{C(x) + F_1(x)\theta^1 + F_2(x)\theta^2 - \psi(\theta^1,\theta^2)\} \tag{1}$$

is said to be a *natural coordinate system*. Here F_1, F_2 and C are randam variables and ψ is a function on the parameter space. The distributions $p(x;\theta^1,\theta^2)$ in (1) are said to be an *exponential family*. For an affine coordinate system, there exists a function ψ on M such that

$$\frac{\partial^2\psi}{\partial\theta^i\partial\theta^j} = g_{ij},$$

where g_{ij} is the Fisher metric with respect to the affine coordinate system. We call ψ the *θ-potential function*. On an exponential family, the θ potential function coincides with ψ in the formula (1).

Proposition 2.2. *Let M be the gamma manifold. Then we have:*

1. *Set $\beta = \nu/\tau$. Then (β, ν) is a natural coordinate system of the 1-connection.*
2. *The function $\psi = \log\Gamma(\nu) - \nu\log\beta$ is the potential function with respect to the natural coordinates (β, ν).*

Proof. Set $\beta = \nu/\tau$. Then gamma distributions can be written as

$$\begin{aligned}
p(t;\beta,\nu) &= \beta^\nu \frac{t^{\nu-1}}{\Gamma(\nu)} e^{-\beta t} \\
&= \exp[-\log t + (\nu\log t - \beta t) - (\log\Gamma(\nu) - \nu\log\beta)]. \qquad (2)
\end{aligned}$$

Hence the set of all gamma distributions is an exponential family. Equality (2) implies that (β, ν) is a natural coordinate system, and $\psi = \log \Gamma(\nu) - \nu \log \beta$ is its potential function. □

For more details, see chapters 2 and 3 in [AN].

Here we give the Fisher metric with respect to natural coordinates (β, ν).

Proposition 2.3. *The Fisher metric with respect to (β, ν) coordinates is given by:*

$$g_{\beta\beta} = \frac{\nu}{\beta^2}, \quad g_{\beta\nu} = g_{\nu\beta} = -\frac{1}{\beta},$$
$$g_{\nu\nu} = \phi'(\nu).$$

□

We remark that there exists also a (-1)-affine coordinate system since the connection $\nabla^{(-1)}$ is flat. Later in this section, we find such a (-1)-affine coordinate system.

In general, for a manifold M with a Riemannian metric g, torsion-free affine connections ∇ and ∇^* mutually dual with respect to g, the tetrad $\{M, g, \nabla, \nabla^*\}$ is said to be a *dually flat space* if ∇ and ∇^* are flat affine connections.

Proposition 2.4 (cf. Theorem 3.6 in [AN])**.** *Let $\{M, g, \nabla, \nabla^*\}$ be a dually flat space. Denote by (θ^i) a ∇-affine coordinate system, and by ψ the θ-potential function. Set $\eta_i = \partial\psi/\partial\theta^i$. Then we have*

1. *The Jacobian matrix of (η_i) with respect to (θ^j) is given by the Riemannian metric g, i.e., $\partial\eta_i/\partial\theta^j = g_{ij}$.*
2. *(η_i) is a ∇^*-affine coordinate system, and (θ^i) and (η_i) are mutually dual with respect to g, i.e., $g(\partial/\partial\theta^i, \partial/\partial\eta_j) = \delta_i^j$.*
3. *$\lambda = \sum \theta^i\eta_i - \psi$ is the η-potential function.*

□

Here we return to the geometry of the gamma manifold.

Since 1- and (-1)-connection on the gamma manifold M are both flat, the tetrad $\{M, g, \nabla^{(1)}, \nabla^{(-1)}\}$ is a dually flat space, so there exist potential functions. As we showed in Proposition 2.2, (β, ν) is a natural coordinate system and $\psi = \log \Gamma(\nu) - \nu \log \beta$ is its potential function. Hence we obtain a (-1)-affine coordinate system and its potential function from Propositions 2.3 and 2.4.

Proposition 2.5. *Let a tetrad $\{M, g, \nabla^{(1)}, \nabla^{(-1)}\}$ be the gamma manifold. Denote by (β, ν) a natural coordinate system on M. Then $(-\nu/\beta, \phi(\nu) - \log \beta)$ is a (-1)-affine coordinate system and*

$$\lambda = -\nu + \nu\phi(\nu) - \log \Gamma(\nu)$$

is the potential function with respect to $(-\nu/\beta, \phi(\nu) - \log \beta)$ coordinates.

We also show that the gamma manifold admits geodesic foliations.

Proposition 2.6. *Let $\{M, g, \nabla^{(1)}, \nabla^{(-1)}\}$ be the gamma manifold. Denote by (β, ν) a natural coordinate system and by $(-\nu/\beta, \phi(\nu) - \log \beta)$ its dual coordinate system. Then, $(-\nu/\beta, \nu)$ and $(\beta, \phi(\nu) - \log \beta)$ are dual geodesic foliations.*

Proof. Since (β, ν) and $(-\nu/\beta, \phi(\nu) - \log \beta)$ are affine coordinate systems, each coordinate curve is a geodesic. From Propositions 2.4 and 2.5, $(-\nu/\beta, \nu)$ and

$(\beta, \phi(\nu) - \log\beta)$ are orthogonal coordinate systems, and they are mutually dual. Then the gamma manifold has geodesic foliations. □

Further to Proposition 2.6, see also Section 3.7 in [AN]. We remark that dual geodesic foliations play an essential role in statistical estimation theory.

3. Affine immersions

In this section, we give the affine immersion which realizes the gamma manifold in $\mathbb{R}^3$.

Let M be an n-dimensional manifold, and f an immersion from M to $\mathbb{R}^n$. Suppose that ξ is a local vector field along f. We say that the pair $\{f, \xi\}$ is an *affine immersion* if the tangent space $T_{f(x)}\mathbb{R}^n = f_*(T_x M) \oplus \mathbb{R}\{\xi_x\}$, where $x \in M$ and $\mathbb{R}\{\xi_x\}$ is the one dimensional subspace spanned by ξ. We call ξ a *transversal vector field*.

Denote by D the standard flat affine connection of $\mathbb{R}^n$. By the decomposition of the tangent space, covariant derivatives appear as follows:

$$D_X f_* Y = f_*(\nabla_X Y) + g(X, Y)\xi, \tag{3}$$

$$D_X \xi = -f_*(SX) + \mu(X)\xi. \tag{4}$$

Then, an affine immersion induces a torsion free affine connection ∇, a symmetric $(0, 2)$ tensor field g, a $(1, 1)$ tensor field S and a 1-form μ on the given manifold M.

Proposition 3.1. *Let M be the gamma manifold with the Fisher metrc g and the exponential connection $\nabla^{(1)}$. Denote by $(\theta^1, \theta^2) = (\beta = \nu/\tau, \nu)$ a natural coordinate system. Then M can be realized in $\mathbb{R}^3$ by the graph of a θ-potential function, namely, M can be realized by the affine immersion $\{f, \xi\}$:*

$$f : \begin{pmatrix} \beta \\ \nu \end{pmatrix} \mapsto \begin{pmatrix} \beta \\ \nu \\ \psi \end{pmatrix}, \quad \xi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where $\psi = \log\Gamma(\nu) - \nu\log\beta$. □

For more details, see Chapter 2 in [NS].

Remark 3.2. Note that the submanifold of exponential distributions, given by $\nu = 1$, is represented in this immersion by the curve

$$(0, \infty) \to \mathbb{R}^3 : \beta \mapsto \{\beta, 1, \log\frac{1}{\beta}\}.$$

4. Divergences

Let $\{M, g, \nabla^{(1)}, \nabla^{(-1)}\}$ be a dually flat space. Denote by (θ^i) the natural coordinate system (1-affine coordinate system) and by (η_i) its dual affine coordinate system. For two points p and p' in M, the function

$$D(p||p') = \psi(p) + \lambda(p') - \sum_{i=1}^{n} \theta^i(p)\eta_i(p') \tag{5}$$

is called a *canonical divergence* or *geometric divergence* (See [AN], [Ku], [Ma-1] and [Ma-2]).

On the other hand, for two probability density functions p and p' on an event space Ω, the function

$$KL(p||p') = \int_\Omega \log \frac{p(x)}{p'(x)} p(x) dx$$

is called a *Kullback-Leibler divergence* or *relative entropy*.

It turns out that we may identify the canonical divergence and the Kullback-Leibler divergence on the gamma manifold.

Proposition 4.1. *Let $\{M, g, \nabla^{(1)}, \nabla^{(-1)}\}$ be the gamma manifold with the Fisher metric, and 1- and (-1)-connections. Then, the canonical divergence D and the Kullback-Leibler divergence KL coincide. For two points $p(t) = p(t;\beta,\nu)$ and $p'(t) = p(t;b,a)$, they are given by*

$$\begin{aligned} D(p||p') &= KL(p||p') \\ &= \log\frac{\Gamma(\nu)}{\Gamma(a)} - \nu\log\frac{\beta}{b} + (a-\nu)\frac{\Gamma'(a)}{\Gamma(a)} - a + \frac{\beta a}{b}. \end{aligned} \tag{6}$$

Proof. From Propositions 2.2 and 2.5, we have the canonical divergence (6) by substituting the definition (5). On the other hand, by straightforward calculation, using Mathematica, we have also the Kullback-Leibler divergence, and it coincides with (6). □

Remark 4.2. We remark that the entropy function S on the gamma manifold is given by

$$S = \nu + (1-\nu)\phi(\nu) + \log\Gamma(\nu) - \log\beta$$

(see [Do-1]). This entropy function contains $-\lambda$, the negative of the potential function. (In the case of the manifold of Gaussian distributions, the entropy function is precisely the negative of the η-potential function, i.e., $S = -\lambda$.) It may be interesting to consider the statistical and geometrical meaning of the difference for gamma manifolds $S - (-\lambda) = \phi(\nu) - \log\beta$.

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