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ALPHA-GEOMETRY OF THE WEIBULL MANIFOLD

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ABSTRACT

The family of Weibull distributions can be considered as a Riemannian manifold, using the Fisher information matrix. In this paper, we derive the α-connections and the α-curvatures. Moreover, we give some illustrative geodesics for the Weibull manifold in three special cases.

KEYWORDS
Weibull distribution, information geometry, $\alpha-$connection, geodesic.

1. INTRODUCTION

Statistical manifolds are representations of smooth families of probability density functions that allow differential geometry to be applied to problems in stochastic processes and information theory.

Amari in 1982 (see [1]) developed the theory of information spaces by introducing the $\alpha-$connection and the $\alpha-$curvatures, and recently, Arwini and Dodson [2] studied the $\alpha-$geometry of some statistical manifolds, namely for the families of the gamma, Gaussian, Mckay bivariate gamma, bivariate Gaussian and the Freund bivariate exponential.

In 1987 Oller [3] derived the differential geometric properties of families of certain extreme value probability distributions, including the Gumbel, Cauchy-Frechet, and Weibull, and he computed the Rao distances between distributions.

2. WEIBULL DISTRIBUTION

The Weibull distribution is one of the most widely used lifetime distributions in reliability engineering, it can be used to represent manufacturing and delivery times in industrial engineering problems.

The Weibull distribution has a probability density function:

$$\mathbf{WE}(x;\theta,\beta)=\beta\theta(\theta x)^{\beta-1}\,e^{-(x\,\theta)^{\beta}}$$

for x≥0, with scale parameter θ>0, and shape parameter β>0.

The Weibull distribution is not of exponential type [1], but it contains the exponential distribution as a special case when $\boldsymbol{\beta}$**=1**. Another special case of the Weibull distribution with $\boldsymbol{\beta}$**=2** is the Rayleigh distribution.

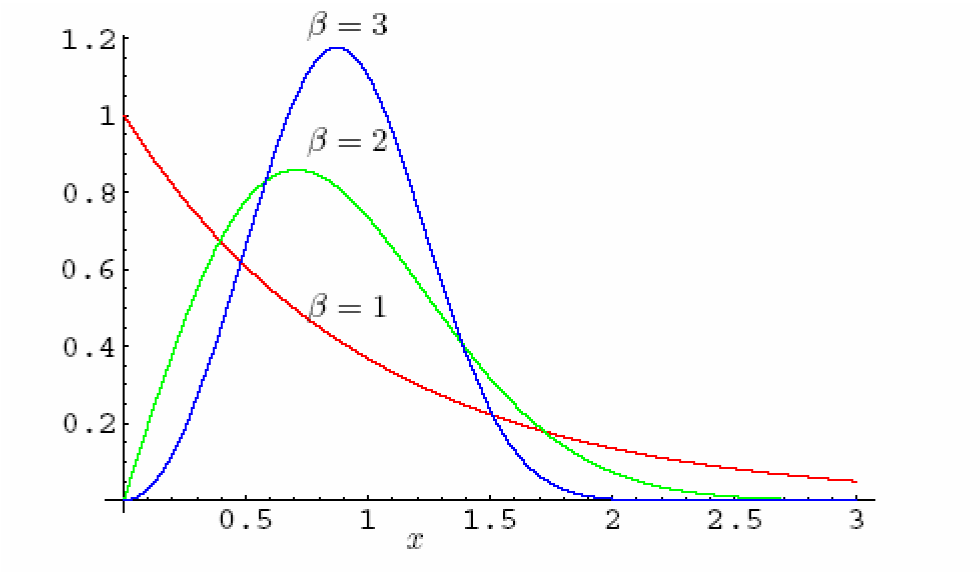

Figure 1: Weibull distributions WE (θ,β) for the range x∈ [0,3]; with parameters θ=1 and β =1,2,3. The case β=1 corresponds to an exponential distribution, β=2 corresponds to a Rayleigh distribution.

3. WEIBULL MANIFOLD

The family of Weibull distributions:

$$M=\left\{\mathbf{WE}/\ \mathbf{WE}(x;\theta,\beta)=\beta\theta(\theta x)^{\beta-1}\,e^{-(x\,\theta)^{\beta}},\ x\geq 0,\ \theta>0,\ \beta>0\right\}$$

is a Riemannian 2-manifold with the Fisher information metric:

$$g=[g_{ij}]=\begin{pmatrix}\frac{\beta^2}{\theta^2} & \frac{1-\gamma}{\theta}\\ \frac{1-\gamma}{\theta} & \frac{6(\gamma-1)^2+\pi^2}{6\beta^2}\end{pmatrix};\quad \text{where } \gamma \text{ is the Euler Gamma}$$

with respect to (θ,β) as a coordinate system.

The α-connection and its curvature tensor and Ricci curvature are all derived, but they have long analytical expressions; here we just mention that the Weibull manifold **M** has a constant α-scalar curvature:

$$R^{(\alpha)}=\frac{12\left(\pi^2\left(3\alpha^2-1\right)-6\alpha^2\left(1+2\zeta(3)\right)\right)}{\pi^4}$$

When α=0, the scalar curvature is negative:

$$R^{(0)} = -\frac{12}{\pi^2}$$

4. GEODESICS

Numerical computations with Mathematica were used to illustrate geodesics and closed curves in the Weibull manifold, in the three special cases β=1,2,3.

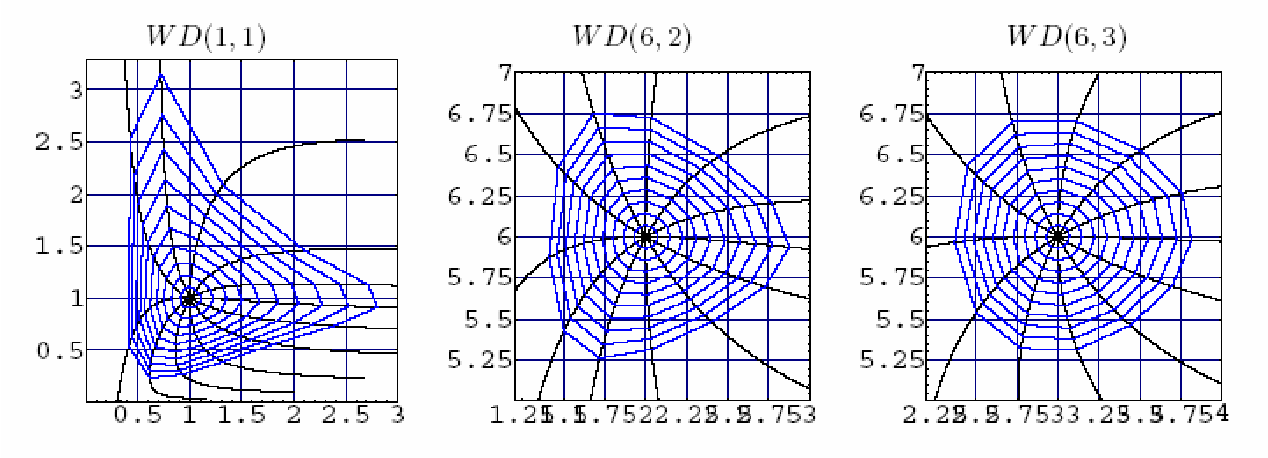

Figure 3: Geodesics and closed curves for Weibull distributions in the cases β=1,2,3.

5. REFERENCES

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