

# A review of some recent work on hypercyclicity\*

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# Abstract

Even linear operators on infinite-dimensional spaces can display interesting dynamical properties and yield important links among functional analysis, differential and global geometry and dynamical systems, with many applications. Hypercyclicity is an essentially infinite-dimensional property, when iterations of the operator generate a dense subspace.

A Fréchet space admits a hypercyclic operator if and only if it is separable and infinite-dimensional. However, semigroups generated by multiples of operators, can yield hypercyclic behaviour on finite dimensional spaces. We review some recent work on hypercyclicity of operators on Banach, Hilbert and Fréchet spaces.

## Introduction

In cases of significance in global analysis [Hamilton 1982, Neeb 2005], physical field theory [Smolentsev 2007], dynamical systems [Bayart, Matheron 2009, Shapiro 2001, Grosse-Erdmann, Manguillot 2011] and finance theory [Emamirad et al. 2011], Banach space representations may break down and we need Fréchet spaces, which have weaker requirements for their topology. Countable products of an infinite-dimensional Banach space are non-normable Fréchet spaces.

Fréchet spaces of sections arise naturally as configurations of a physical field where the moduli space, consisting of inequivalent configurations of the physical field, is the quotient of the infinite-dimensional configuration space  $\mathcal{X}$  by the appropriate symmetry gauge group. Typically,  $\mathcal{X}$  is modelled on a Fréchet space of smooth sections of a vector bundle over a closed manifold.

An important case is the metric geometry of the Fréchet manifold of all  $C^\infty$  Riemannian metrics on a fixed closed finite-dimensional orientable manifold [Smolentsev 2007, Clarke 2010].

Another concerns Sobolev metrics and geodesic behaviour on groups of diffeomorphisms of a finite-dimensional manifold under the condition that the diffeomorphisms decay suitably rapidly to the identity [Micheli et al. 2012].

Lie-Fréchet groups of diffeomorphisms of closed Riemannian manifolds arise as ILH-manifolds, that is as inverse (i.e. projective) limits of Hilbert manifolds [Omori 1970, Omori 1997].

Unlike Fréchet manifolds, Hilbert manifolds do support the main theorems of calculus.

Fréchet spaces lack a general solvability theory of differential equations, even linear ones; also, the space of continuous linear mappings drops out of the category while the space of linear isomorphisms does not admit a reasonable Lie group structure.

Such shortcomings can be worked round to a certain extent by representing Fréchet spaces as projective limits of Banach spaces and in the manifold cases by requiring the geometric structures to carry through as projective limits [Galanis 1997, Vassiliou, Galanis 1997, Dodson, Galanis 2004], [Dodson et al. 2005, Dodson et al. 2006], and a recent survey [Dodson 2012].

Lie group actions on Banach spaces, details of solutions of differential equations on each step of a projective limit and integration of some Lie algebras of vector fields have been elaborated in a detailed study [Walter 2010].

An open problem is the extension to Banach, Hilbert and Fréchet bundles of the results on projection and lifting of harmonicity for tangent, second tangent and frame bundles obtained with Vazquez-Abal [Dodson, Vazquez-Abal 1990, Dodson, Vazquez-Abal 1992], for finite-dimensional Riemannian manifolds:

$$\begin{array}{ccccc}
 (FM, Fg) & \xrightarrow{\pi_{FM}} & (M, g) & \xleftarrow{\pi_{TM}} & (TM, Tg) \\
 \downarrow Ff & & \downarrow f & & \downarrow Tf \\
 (FN, Fh) & \xrightarrow{\pi_{FN}} & (N, h) & \xleftarrow{\pi_{TN}} & (TN, Th)
 \end{array}$$

In this diagram  $f$  needs to be a local diffeomorphism of Riemannian manifolds for the frame bundle morphism  $Ff$  to be defined. **It was shown that  $Ff$  is totally geodesic if and only if  $f$  is totally geodesic; when  $f$  is a local diffeomorphism of flat manifolds then  $Ff$  is harmonic if  $f$  is harmonic. Also, the diagonal map  $\pi_{FN} \circ Ff = f \circ \pi_{FM}$  is harmonic if and only if  $f$  is harmonic, and  $Ff$  is harmonic if and only if  $Tf$  is**

The corresponding result for the tangent bundle projection:  $Tf$  is totally geodesic if and only if  $f$  is totally geodesic had already been established.

**It follows that  $\pi_{TM}$  is a harmonic Riemannian submersion and the diagonal map  $\pi_{TN} \circ Tf = f \circ \pi_{TM}$  is harmonic if and only if  $f$  is harmonic [Dodson, Vazquez-Abal 1992, Smith 1975].**

It would be interesting to extend the above to the infinite dimensional case of an inverse limit Hilbert (ILH) manifold  $\mathbb{E} = \lim_{\infty \leftarrow s} \mathbb{E}^s$ , of a projective system of smooth Hilbert manifolds  $\mathbb{E}^s$ , consisting of sections of a tensor bundle over a smooth compact finite dimensional Riemannian manifold  $(M, g)$ .

Such spaces arise in geometry and physical field theory with many desirable properties but it is necessary to establish existence of the projective limits for various geometric objects.

Via the volume form on ( $n$ -dimensional compact)  $(M, g)$  a weak induced metric on the space of tensor fields is  $\int_M g(X, Y)$  but there is a stronger family [Smolentsev 2007] of inner products on  $\mathbb{E}^s$ , the completion Hilbert space of sections. For sections  $X, Y$  of the given tensor bundle over  $M$  we put

$$(X, Y)_{g,s} = \sum_{i=0}^s \int_M g(\nabla^{(i)} X, \nabla^{(i)} Y) \quad s \geq 0. \quad (1)$$

**Then the limit  $\mathbb{E} = \lim_{\infty \leftarrow s} \mathbb{E}^s$  with limiting inner product  $g_{\mathbb{E}}$  is a Fréchet space with topology independent of the choice of metric  $g$  on  $M$ . It is known that the smooth diffeomorphisms  $f : (M, g) \rightarrow (M, g)$  form a strong ILH-Lie group  $Diff_M$  modelled on the ILH manifold**

$$\Gamma(TM) = \lim_{\infty \leftarrow s} \Gamma^s(TM)$$

**of smooth sections of the tangent bundle. Moreover, the curvature and Ricci tensors are equivariant under the action of  $Diff_M$ , which yields the Bianchi identities as consequences.**



The diagram of Hilbert manifolds of sections of vector bundles over smooth compact finite dimensional Riemannian manifolds  $(M, g)$ ,  $(N, h)$  with  $\mathbb{E} = \Gamma(TM)$ ,  $\mathbb{F} = \Gamma(TN)$ . Diagonal lift metrics are induced via the horizontal-vertical splittings defined by the Levi-Civita connections  $\nabla^g, \nabla^h$  on the base manifolds (cf. [Sasaki 1958, Dombrowski 1962, Kowalski 1978]), effectively applying the required evaluation to corresponding projections; we abbreviate these to

$$Tg_{\mathbb{E}} = (g_{\mathbb{E}}, g_{\mathbb{E}}), \quad Th_{\mathbb{F}} = (h_{\mathbb{F}}, h_{\mathbb{F}}),$$

$$\begin{array}{ccc} (T\mathbb{E}, Tg_{\mathbb{E}}) & \xrightarrow{T\phi} & (T\mathbb{F}, Th_{\mathbb{F}}) \\ \pi_{T\mathbb{E}} \downarrow & & \downarrow \pi_{T\mathbb{F}} \\ (\mathbb{E}, g_{\mathbb{E}}) & \xrightarrow{\phi} & (\mathbb{F}, h_{\mathbb{F}}) \end{array}$$

For example: **a smooth map of Riemannian manifolds  $f : (M, g) \rightarrow (N, h)$  defines a fibre preserving map  $f^*$  between their tensor bundles and induces such a smooth map  $\phi$  between the spaces of sections.**

The Laplacian  $\Delta$  on our Hilbert manifold  $\mathbb{E}$  is defined by  $\Delta = -\operatorname{div} \nabla^{\mathbb{E}} d$  where the generalized divergence  $-\operatorname{div}$  is the trace of the covariant derivation operator  $\nabla^{\mathbb{E}}$ , so  $\operatorname{div}$  is the adjoint of the covariant derivation operator  $\nabla^{\mathbb{E}}$ .

At this juncture we defer to future studies the open problems of lifting and projection of harmonicity in ILH manifolds and turn to the characterization of linear operators then review work reported in the last few years on the particular property of hypercyclicity, when iterations generate dense subsets.

## Dynamics of linear operator equations

A common problem in applications of linear models is the characterization and solution of continuous linear operator equations on Hilbert, Banach and Fréchet spaces.

However, there are many open problems. For example it is known that:

for a continuous linear operator  $T$  on a Banach space  $\mathbb{E}$  there is no non-trivial closed subspace nor non-trivial closed subset  $A \subset \mathbb{E}$  with  $TA \subset A$ , but this is an unsolved problem on Hilbert and Fréchet spaces [Martin 2011, Banos 2011].

There has been substantial interest from differential geometry and dynamical systems in **hypercyclic** operators, whose iterations generate dense subsets. In this survey we look at some of the results on hypercyclicity of operators that have been reported in the last few years.

Continuous linear operator  $T$  on a topological vector space  $\mathbb{E}$  has a **periodic point**  $f \in \mathbb{E}$  if, for some  $n \in \mathbb{N}$  we have  $T^n f = f$ .  $T$  is **cyclic** if for some  $f \in \mathbb{E}$  the span of  $\{T^n f, n \geq 0\}$  is dense in  $\mathbb{E}$ . On finite-dimensional spaces there are many cyclic operators but no hypercyclic operators.

## Hypercyclicity properties

Continuous linear operator  $T$  on a topological vector space  $\mathbb{E}$  is **hypercyclic** if, for some  $f \in \mathbb{E}$ , called a **hypercyclic vector**, the set  $\{T^n f, n \geq 0\}$  is dense in  $\mathbb{E}$ , and **supercyclic** if the projective space orbit  $\{\lambda T^n f, \lambda \in \mathbb{C}, n \geq 0\}$  is dense in  $\mathbb{E}$ . These properties are called **weakly hypercyclic**, **weakly supercyclic** respectively, if  $T$  has the property in the weak topology—the smallest making every member of the dual space continuous.

If  $T$  is invertible, then it is hypercyclic if and only if  $T^{-1}$  is hypercyclic. It is known for  $\ell^p(\mathbb{N})$ , the Banach space of complex sequences with  $p$ -summable modulus  $p \geq 1$  and backward shift operator  $B_{-1} : (x_0, x_1, x_2, \dots) \mapsto (x_1, x_2, x_3, \dots)$ , that  $\lambda B_{-1}$  is hypercyclic on  $\ell^p(\mathbb{N})$  if and only if  $|\lambda| > 1$ .

[De La Rosa 2011] proved the following for a weak hypercyclic  $T$  :

- (i)  $T \oplus T$  need not be weakly hypercyclic, with an example on  $\ell^p(\mathbb{N}) \oplus \ell^p(\mathbb{N})$ ,  $1 \leq p < \infty$
- (ii)  $T^n$  is weakly hypercyclic for every  $n > 1$
- (iii) For all unimodular  $\lambda \in \mathbb{C}$ , we have  $\lambda T$  weakly hypercyclic.

A weakly hypercyclic operator has many of the same properties as a hypercyclic operator: **its adjoint has no eigenvalue and every component of its spectrum must intersect the unit circle.**

On every separable Banach space, hypercyclicity is equivalent to transitivity: **for every pair of nonempty, norm open sets  $(U, V)$ , we have  $T^n(U) \cap V \neq \emptyset$  for some  $n \in \mathbb{N}$ .**

**On the Fréchet space of analytic functions on  $\mathbb{C}^N$  every linear partial differential operator with constant coefficients and positive order has a hypercyclic vector.** However, that proof does not carry over to the weak topology.

On a separable infinite-dimensional complex Hilbert space  $\mathbb{H}$  the set of hypercyclic operators is dense in the strong operator topology, and moreover the linear span of hypercyclic operators is dense in the operator norm topology. The non-hypercyclic operators are dense in the set of bounded operators  $B(\mathbb{H})$  on  $\mathbb{H}$ , but the hypercyclic operators are not dense in the complement of the closed unit ball of  $B(\mathbb{H})$ , [Chan 2001].

For linear operators acting on a reflexive Banach space  $\mathbb{E}$  with the weak topology, a bounded operator, transitive on an open bounded subset of  $\mathbb{E}$  with the weak topology, is weakly hypercyclic, [Rezai 2011].

If a linear operator is hypercyclic, then having a hypercyclic vector yields a dense subspace with all nonzero vectors hypercyclic. A **hypercyclic subspace** for a linear operator is an infinite-dimensional closed subspace all of whose nonzero vectors are hypercyclic.

On the Fréchet space  $\mathbb{H}(\mathbb{C})$  of functions analytic on  $\mathbb{C}$  : **the translation by a fixed nonzero  $\alpha \in \mathbb{C}$  is hypercyclic and so is the differentiation operator  $f \mapsto f'$ .**

**All infinite-dimensional separable Banach spaces admit hypercyclic operators.**

On the other hand, **no finite-dimensional Banach space admits a hypercyclic operator [Ansari 1997].**

**Every nonzero power  $T^m$  of a hypercyclic linear operator  $T$  is hypercyclic. Backward weighted shifts on  $\ell^2$  such that  $T(e_i) = w_i e_{i-1}$  ( $i \geq 1$ ) and  $T(e_0) = 0$  with positive  $w_i$  were used to show that  $T + I$  is hypercyclic [Salas 1995].**

**A Fréchet space admits a hypercyclic operator if and only if it is separable and infinite-dimensional and the spectrum of a hypercyclic operator must meet the unit circle.**



A **sequence** of linear operators  $\{T_n\}$  on  $\mathbb{E}$  is hypercyclic if  $\exists f \in \mathbb{E} : \{T_n f, n \in \mathbb{N}\} \subseteq \mathbb{E}$  is dense [Chen, Shaw 2006].

It satisfies the **Hypercyclicity Criterion** for an increasing sequence  $\{n(k)\} \subset \mathbb{N}$  if there are dense subsets  $X_0, Y_0 \subset \mathbb{E}$  satisfying:

(i)  $(\forall f \in X_0) T_{n(k)} f \rightarrow 0$

(ii)  $(\forall g \in Y_0) \exists \{u(k)\} \subset \mathbb{E}$  such that  $u(k) \rightarrow 0$  and  $T_{n(k)} u(k) \rightarrow 0$ .

**On a separable Fréchet space  $\mathbb{F}$  a continuous linear operator  $T$  satisfies the Hypercyclicity Criterion if and only if  $T \oplus T$  is hypercyclic on  $\mathbb{F} \oplus \mathbb{F}$ . Moreover, if  $T$  satisfies the Hypercyclicity Criterion then so does every power  $T^n$  for  $n \in \mathbb{N}$ , [Bès, Peris 1999].**

A vector  $x$  is **universal** for operators  $(T_n : n \in \mathbb{N})$  on a Banach space  $\mathbb{E}$  if  $\{T_n x : n \in \mathbb{N}\}$  is dense;  $x$  is **frequently universal** if for each non-empty open set  $U \subset \mathbb{E}$  the set  $K = \{n : T_n x \in U\}$  has positive lower density, i.e.

$$\liminf_{N \rightarrow \infty} |\{n \leq N : n \in K\}|/N > 0.$$

A **frequently hypercyclic** vector of  $T$  is such that, for each non-empty open set  $U$ , the set  $\{n : T^n \in U\}$  has positive lower density, a stronger requirement than hypercyclicity.

[Bonilla, Grosse-Erdmann 2007] extended a sufficient condition for frequent hypercyclicity from [Bayart, Grivaux 2006], to frequent universality. [Beise 2012] extended this work and gave a sufficient condition for frequent universality in the Fréchet case.

Let  $\varphi$  be a holomorphic self-map of  $B_N$ , the open unit ball of  $\mathbb{C}^N$ . Then the composition operator with symbol  $\varphi$  is  $C_\varphi : f \mapsto f \circ \varphi$  for  $f \in H(B_N)$  the space of holomorphic maps on  $B_N$ . The multiplication operator induced by  $\psi \in H(B_N)$  is  $M_\psi(f) = \psi \cdot f$  and the weighted composition operator induced by  $\psi, \varphi$  is  $W_{\psi, \varphi} = M_\psi C_\varphi$ .

Extending the work [Yousefi, Rezaei 2007], [Chen, Zhou 2011] obtained necessary and sufficient conditions for the hypercyclicity of weighted composition operators (cf. also [Bonet, Domański 2011]) acting on the complete vector space of holomorphic functions on the open unit ball  $B_N$  of  $\mathbb{C}^N$ .

**If  $C_\varphi$  is hypercyclic, then so is  $\lambda C_\varphi$  for all unimodular  $\lambda \in \mathbb{C}$ ; also, if  $\varphi$  has an interior fixed point  $w$  and  $\psi \in H(B_N)$  satisfies**

$$|\psi(w)| < 1 < \liminf_{|z| \rightarrow 1} |\psi(z)|,$$

**then the adjoint  $W_{\psi, \varphi}^*$  is hypercyclic.**

Hypercyclic composition operators  $C_\varphi : f \mapsto f \circ \varphi$  were characterized on the space of functions holomorphic on  $\Omega \subset \mathbb{C}^N$ , a pseudoconvex domain with  $\varphi$  a holomorphic self-mapping of  $\Omega$ . In the case when all the balls are relatively compact in  $\Omega$ , for simply connected or infinitely connected planar domains, hypercyclicity of  $C_\varphi$  implies it is **hereditarily hypercyclic** (i.e.  $C_\varphi \oplus C_\varphi$  is hypercyclic), [Zajac 2012].

The Volterra composition operators  $V_\varphi$  for  $\varphi$  a measurable self-map of  $[0, 1]$  on functions  $f \in L^p[0, 1]$ ,  $1 \leq p \leq \infty$

$$(V_\varphi f)(x) = \int_0^\varphi(x) f(t) dt \quad (2)$$

generalize the classical Volterra operator  $V$ , the case when  $\varphi$  is the identity.  $V_\varphi$  is measurable, and compact on  $L^p[0, 1]$ , [Montes-Rodríguez et al, 2011].

Consider the Fréchet space  $\mathbb{F} = C_0[0, 1)$ , of continuous functions vanishing at zero with the topology of uniform convergence on compact subsets of  $[0, 1)$ . **The action of  $V_\varphi$  on  $C_0[0, 1)$  is hypercyclic when  $\varphi(x) = x^b$ ,  $b \in (0, 1)$ .** This result has now been extended to give the following complete characterization.

**Theorem ([Montes-Rodríguez et al, 2011])**

**For  $\varphi \in C_0[0, 1)$  the following are equivalent**

- (i)  $\varphi$  is strictly increasing with  $\varphi(x) > x$  for  $x \in (0, 1)$**
- (ii)  $V_\varphi$  is weakly hypercyclic**
- (iii)  $V_\varphi$  is hypercyclic.**

Furthermore, for every strictly increasing  $\varphi$  with  $\varphi(x) < x$ ,  $x \in (0, 1]$  that  $V_\varphi$  is supercyclic and  $I + V_\varphi$  is hypercyclic when  $V_\varphi$  acts on  $L^p[0, 1]$ ,  $p \geq 1$ , or on  $C_0[0, 1]$ .

The conjugate set  $\{L^{-1}TL : L \text{ invertible}\}$  of any supercyclic operator  $T$  on a separable, infinite dimensional Banach space contains a path of supercyclic operators which is dense with the strong operator topology, and the set of common supercyclic vectors for the path is a dense  $G_\delta$  set (countable intersection of open and dense sets) if  $\sigma_p(T^*)$  is empty, [Shu et al. 2011] .

**For  $H_{bc}(\mathbb{E})$ , the space of bounded functions on compact subsets of Banach space  $\mathbb{E}$ , when  $\mathbb{E}$  has separable dual  $\mathbb{E}^*$  then for nonzero  $\alpha \in \mathbb{E}$ ,  $T_\alpha : f(x) \mapsto f(x + \alpha)$  is hypercyclic, [Karami et al. 2011].**

As for other cases of hypercyclic operators on Banach spaces, it would be interesting to know when the property persists to projective limits of the domain space.

Let  $T$  be a continuous linear operator on an infinite dimensional Hilbert space  $\mathbb{H}$  and let left multiplication be hypercyclic with respect to the strong operator topology. Then there exists a Fréchet space  $\mathbb{F}$  containing  $\mathbb{H}$ , where  $\mathbb{F}$  is the completion of  $\mathbb{H}$ , and for every nonzero vector  $f \in \mathbb{H}$  the orbit  $\{T^n f, n \geq 0\}$  meets any open subbase of  $\mathbb{F}$ , [Yousefi, Ahmadian 2009]

**The direct sum of two hypercyclic operators need not be hypercyclic and even the direct sum of a hypercyclic operator with itself  $T \oplus T$  need not be hypercyclic, [De La Rosa, Read 2009].**

There is a linear operator  $T$  such that the direct sum  $T \oplus T \oplus \dots \oplus T = T^{\oplus m}$  of  $m$  copies of  $T$  is a hypercyclic operator on  $\mathbb{F}^m$  for each  $m \in \mathbb{N}$ , [Shkarin 2010b].

Consider the space  $H(\mathbb{U})$  of holomorphic functions on  $\mathbb{U}$ , the open unit disc in  $\mathbb{C}$ . Each  $\phi \in H(\mathbb{U})$  and holomorphic self-map  $\psi$  of  $\mathbb{U}$  induce a weighted linear operator  $C_{\phi, \psi}$  sending  $f(z)$  to  $\phi(z)f(\psi(z))$ .

This property includes both composition  $C_{\psi}$ , ( $\phi = 1$ ) and multiplication  $M_{\phi}$ , ( $\psi = 1$ ) as special cases. Any nonzero multiple of  $C_{\psi}$  is chaotic on  $H(\mathbb{U})$  if  $\psi$  has no fixed point in  $\mathbb{U}$ , [Rezai 2011b].



An  $m$ -tuple  $(T, T, \dots, T)$  is called **disjoint hypercyclic** if there exists  $f \in \mathbb{F}$  such that  $(T_1^n f, T_2^n f, \dots, T_m^n f)$  is dense in  $\mathbb{F}^m$ .

Characterization has been made of disjoint hypercyclicity and disjoint supercyclicity of finitely many linear fractional composition operators acting on spaces of holomorphic functions on the unit disc, [Bès et al. 2011].

Finitely many hypercyclic composition operators  $f \mapsto f \circ \varphi$  on the unit disc  $\mathbb{D}$  generated by non-elliptic automorphisms  $\varphi$  need not be disjoint nor need they be so on the Hardy space  $H^2(\mathbb{D})$  of square-summable power series on the unit disc where

$$H^2(\mathbb{D}) = \left\{ f = z \mapsto \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}) : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}$$

**Countable products of copies of an infinite-dimensional Banach space are examples of non-normable Fréchet spaces that do not admit a continuous norm.**

[Albanese 2011] showed that For  $\mathbb{F}$  a separable, infinite-dimensional real or complex Fréchet space admitting a continuous norm and  $\{v_n \in \mathbb{F} : n \geq 1\}$  a dense set of linearly independent vectors, there exists a continuous linear operator  $T$  on  $\mathbb{F}$  such that the orbit under  $T$  of  $v_1$  is exactly the set  $\{v_n : n \geq 1\}$ .

This extended a result for Banach spaces to the setting of non-normable Fréchet spaces that do admit a continuous norm, cf. [Grivaux 2003].

## **Semigroups and $n$ -tuples of operators**

**A Fréchet space admits a hypercyclic operator if and only if it is separable and infinite-dimensional. No finite-dimensional Banach space admits a hypercyclic operator, but a finitely-generated semigroup of operators instead of a single operator can generate a dense subset.**

**For semigroups of operators  $\Gamma = \langle T_1, T_2, \dots, T_k \rangle$  on a finite dimensional vector space over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$  :  $\Gamma$  is hypercyclic or topologically transitive if there exists  $x \in \mathbb{K}^n$  such that  $\{Tx : T \in \Gamma\}$  is dense in  $\mathbb{K}^n$ , [Javaheri 2011].**

**Examples exist of  $n \times n$  matrices  $A$  and  $B$  such that almost every column vector had an orbit under the action of the semigroup  $\langle A, B \rangle$  dense in  $\mathbb{K}^n$ , and in every finite dimension there are pairs of commuting matrices which form a locally hypercyclic but non-hypercyclic tuple [Costakis et al. 2009, Costakis et al. 2010].**

In the non-abelian case, there exists a 2-generator hypercyclic semigroup in any dimension in both real and complex cases, [Javaheri 2011b].

**Thus there exists a dense 2-generator semigroup in any dimension in both real and complex cases. Since powers of a single matrix can never be dense, this result is optimal.**

The minimal number of matrices on  $\mathbb{C}^n$  required to form a hypercyclic abelian semigroup on  $\mathbb{C}^n$  is  $n + 1$ , and the action of any abelian semigroup finitely generated by matrices on  $\mathbb{C}^n$  or  $\mathbb{R}^n$  is never  $k$ -transitive for  $k \geq 2$ , [Ayadi 2011].

There are hypercyclic  $(n + 1)$ -tuples of diagonal matrices on  $\mathbb{C}^n$ , but there are no hypercyclic  $n$ -tuples of diagonalizable matrices on  $\mathbb{C}^n$ , [Feldman 2008].

**The minimal cardinality of a hypercyclic tuple of operators is  $n + 1$  on  $\mathbb{C}^n$  and  $\frac{n}{2} + \frac{5+(-1)^n}{4}$  on  $\mathbb{R}^n$ , [Shkarin 2011b].**

There are non-diagonalizable tuples of operators on  $\mathbb{R}^2$  which possess an orbit that is neither dense nor nowhere dense and a hypercyclic 6-tuple of operators on  $\mathbb{C}^3$  such that every operator commuting with each member of the tuple is non-cyclic.

Every infinite-dimensional separable complex (real) Fréchet space admits a hypercyclic 6-tuple (4-tuple)  $T$  of operators such that there are no cyclic operators commuting with  $T$ .

**Moreover, every hypercyclic tuple  $T$  on  $\mathbb{C}^2$  or  $\mathbb{R}^2$  contains a cyclic operator.**

An operator is called **mixing** if for all nonempty open subsets  $U, V$ , there is  $n \in \mathbb{N}$  such that  $T^n(U) \cap V \neq \emptyset$  for each  $n \geq m$ .

Every hypercyclic operator is transitive, since it has a vector with a dense orbit. If the space is complete separable and metrizable, then the converse implications hold: every transitive operator is hypercyclic and every mixing operator is hereditarily hypercyclic.

**A continuous linear operator on a topological vector space with weak topology is mixing if and only if its dual operator has no finite dimensional invariant subspaces, [Shkarin 2011].**

There exist hypercyclic strongly continuous holomorphic groups of operators containing non-hypercyclic operators. Also, examples where a family of hypercyclic operators has no **common** hypercyclic vector, an important property in linear dynamics, [Bayart 2011].

There is a complete characterization of abelian subgroups of  $GL(n, \mathbb{R})$  with a locally dense (resp. dense) orbit in  $\mathbb{R}^n$ . For finitely generated subgroups, this characterization is explicit and it is used to show that no abelian subgroup of  $GL(n, \mathbb{R})$  generated by the integer part of  $(n + 1/2)$  matrices can have a dense orbit in  $\mathbb{R}$ , [Ayadi et al. 2011].

It is explicit also for finitely generated abelian affine groups. In particular no abelian group generated by  $n$  affine maps on  $\mathbb{C}^n$  has a dense orbit. An example is given of a group with dense orbit in  $\mathbb{C}^2$ , [Ayadi 2011b].

**The minimal cardinality of a hypercyclic tuple of operators on  $\mathbb{C}^n$  (respectively, on  $\mathbb{R}^n$ ) is  $n + 1$  (respectively,  $\frac{n}{2} + \frac{5+(-1)^n}{4}$ )**  
There are non-diagonalizable tuples of operators on  $\mathbb{R}^2$  which possess an orbit being neither dense nor nowhere dense and construct a hypercyclic 6-tuple of operators on  $\mathbb{C}^3$  such that every operator commuting with each member of the 6-tuple is non-cyclic, [Shkarin 2011b].

The Black-Scholes equation, used (and sometimes misused! [Stewart 2012]) for the value of a stock option, yields a semigroup on spaces of continuous functions on  $(0, \infty)$  that are allowed to grow at both 0 and  $\infty$ , which is important since the standard initial value is an unbounded function.

There is a family of Banach spaces, parametrized by two market properties on some ranges of which the Black-Scholes semigroup is strongly continuous and chaotic, [Emamirad et al. 2011].



## Topological transitivity and mixing

An operator  $T$  is hypercyclic on a separable Fréchet space  $\mathbb{F}$  if it has the **topological transitivity property**:  $\forall$  nonempty open subsets  $U, V \subseteq \mathbb{F} \exists n \in \mathbb{N}$  such that  $T^n(U) \cap V \neq \emptyset$ , [Grosse-Erdmann 1999].

If  $T^n$  satisfies the Hypercyclicity Criterion then  $T$  is **topologically mixing** in the sense that: for every pair of nonempty open subsets  $U, V \subseteq \mathbb{F}$  there is some  $N \in \mathbb{N}$  such that  $T^n(U) \cap V \neq \emptyset$  for all  $n \geq N$ , [Costakis, Sambarino 2003, Chen, Shaw 2006].

There is a characterization of disjoint hypercyclicity and disjoint supercyclicity for linear fractional composition operators  $C_\varphi : f \mapsto f \circ \varphi$  on  $\nu$ -weighted Hardy spaces  $S_\nu$ ,  $\nu \in \mathbb{R}$ , of analytic functions on the unit disc, [Bès et al. 2011]:

$$S_\nu = \left\{ f = z \mapsto \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}) : \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 (n+1)^{2\nu} < \infty \right\}$$

It was known that a linear fractional composition operator  $C_\varphi$  is hypercyclic on  $S_\nu$  if and only if  $\nu < \frac{1}{2}$  and  $C_\varphi$  is hypercyclic on  $H^2(\mathbb{D}) = S_0$ . Also, if  $\nu < \frac{1}{2}$  then  $C_\varphi$  is supercyclic on  $S_\nu$  if and only if it is hypercyclic on  $S_\nu$ . This is now extended to the projective limit of  $\{S_\nu : \nu < \frac{1}{2}\}$ , [Bès et al. 2011].

[Shkarin 2011] proved that:

**A continuous linear operator  $T$  on a topological vector space with weak topology is mixing if and only if its dual operator has no finite-dimensional invariant subspace. Hence for every hypercyclic  $T$  on the countable product of copies of  $\mathbb{K} = \mathbb{C}$  or  $\mathbb{R}$ , we have also that  $T \oplus T$  is hypercyclic.**

There is a class of topological vector spaces admitting a mixing uniformly continuous operator group  $\{T_t\}_{t \in \mathbb{C}^n}$  with holomorphic dependence on the parameter  $t$ , and a class of topological vector spaces admitting no supercyclic strongly continuous operator semigroups  $\{T_t\}_{t \geq 0}$ , [Shkarin 2011].

## Subspace hypercyclicity

A continuous linear operator  $T$  on a Hilbert space  $\mathbb{H}$  is  **$M$ -hypercyclic** for a subspace  $M$  of  $\mathbb{H}$  if there exists a vector such that the intersection of its orbit and  $M$  is dense in  $M$ . For example, if  $T$  is subspace-hypercyclic, then its spectrum must intersect the unit circle, but not every element of the spectrum need do so.

**Subspace-hypercyclicity is a strictly infinite-dimensional phenomenon; neither compact operators nor hyponormal (i.e.  $\|Tx\| \geq \|T^*x\|, \forall x \in \mathbb{H}$ ) bounded operators are subspace-hypercyclic, [Madore, Martinez-Avendano 2011].**

For closed  $M$  in separable Banach  $\mathbb{E}$ ,  $M$ -hypercyclicity is implied by  $M$ -transitivity—i.e. for all disjoint nonempty open subsets  $U, V$  of  $M$  there is a number  $n$  such that  $U \cap T^{-n}V$  contains a nonempty open set of  $M$ . However,  **$M$ -hypercyclicity need not imply  $M$ -transitivity, [Le 2011].**

The (weakly) topological transitivity of semigroup  $S$  of bounded linear operators on a real Banach space is the property for all nonempty (weakly) open sets  $U, V \exists T \in S : TU \cap V \neq \emptyset$ . This was characterized for the families of operators  $\{S^t | t \in \mathbb{N}\}$ ,  $\{kS^t | t \in \mathbb{N}, k > 0\}$ , and  $\{kS^t | t \in \mathbb{N}, k \in \mathbb{R}\}$ , in terms of the point spectrum of the dual operator  $S^*$ .

Unlike topological transitivity in the norm topology, (=hypercyclicity) with concomitant highly irregular behaviour of the semigroup, quite good behaviour of weakly topologically transitive semigroups, [Desch, Schappacher 2011].

**An example is the positive-definite bounded self-adjoint  $S : L^2([0, 1]) \rightarrow L^2([0, 1]) : u(\xi) \mapsto \frac{u(\xi)}{\xi+2}$ .**

**Then  $S = S^*$  and has empty point spectrum so  $\{S^t : t \in \mathbb{N}\}$  is weakly topologically transitive but cannot be weakly hypercyclic because  $S^t \rightarrow 0$  in operator norm if  $t \rightarrow \infty$ .**

A weakly open set in an infinite-dimensional Banach space contains a subspace of finite codimension but a 'small' neighborhood has many large vectors, easily hit by trajectories.

If  $x$  is a hypercyclic vector for  $T$ , then so is  $T^n x$  for all  $n \in \mathbb{N}$ , and  $T^n x$  is in the range of  $T$ . Since the range of  $T$  is dense, one might expect that most if not all of an operator's hypercyclic vectors lie in its range.

However,

**for every non-surjective hypercyclic operator  $T$  on a Banach space, the set of hypercyclic vectors for  $T$  that are not in its range is large, in that it is not expressible as a countable union of nowhere dense sets. This provides also a sense by which the range of an arbitrary hypercyclic operator  $T$  is large in its set of hypercyclic vectors for  $T$ , [Rion 2011].**

## Chaotic behaviour

A continuous linear operator  $T$  on a topological vector space  $\mathbb{E}$  has a periodic point  $f \in \mathbb{E}$  if, for some  $n \in \mathbb{N}$  we have  $T^n f = f$ .

$T$  is cyclic if for some  $f \in \mathbb{E}$  the span of  $\{T^n f, n \geq 0\}$  is dense in  $\mathbb{E}$ . On finite-dimensional spaces there are many cyclic operators but no hypercyclic operators.

$T$  is called chaotic if it is hypercyclic and its set of periodic points is dense in  $\mathbb{E}$ . Each operator on the Fréchet space of analytic functions on  $\mathbb{C}^N$ , which commutes with all translations and is not a scalar multiple of the identity, is chaotic, [Godefroy, Shapiro 1991, Grosse-Erdmann, Manguillot 2011].

Weighted composition operators on the space  $H(\mathbb{U})$  of holomorphic functions on  $\mathbb{U}$ , the open unit disc in  $\mathbb{C}$  are such that each  $\phi \in H(\mathbb{U})$  and holomorphic self-map  $\psi$  of  $\mathbb{U}$  induce a weighted linear operator  $C_{\phi, \psi} : f(z) \mapsto \phi(z)f(\psi(z))$ . Every nonzero multiple of  $C_{\psi}$  is chaotic on  $H(\mathbb{U})$  if  $\psi$  has no fixed point in  $\mathbb{U}$ , [Rezai 2011b].

There is a family of Banach spaces, parametrized by two market properties on some ranges of which the Black-Scholes semigroup is strongly continuous and chaotic, [Emamirad et al. 2011].

The conjugate set  $\mathcal{C}(T) = \{L^{-1}TL : L \text{ invertible}\}$  of a hypercyclic operator  $T$  consists entirely of hypercyclic operators, dense in the algebra of bounded linear operators in the strong operator topology. **On an infinite-dimensional Hilbert space, there is a path of chaotic operators, dense in the operator algebra with the strong operator topology, and along which every operator has the exact same dense  $G_\delta$  set of hypercyclic vectors, [Chan, Sanders 2011].**

The conjugate set of any hypercyclic operator on a separable, infinite dimensional Banach space always contains a path of operators which is dense with the strong operator topology, and yet the set of common hypercyclic vectors for the entire path is a dense  $G_\delta$  set. As a corollary, **the hypercyclic operators on such a Banach space form a connected subset of the operator algebra with the strong operator topology,**

Originally defined on a metric space  $(X, d)$ , a **Li-Yorke chaotic map**  $f : X \rightarrow X$  is such that there exists an uncountable subset  $\Gamma \subset X$  in which every pair of distinct points  $x, y$  satisfies  $\liminf_n d(f^n x, f^n y) = 0$  and  $\limsup_n d(f^n x, f^n y) > 0$  then  $\Gamma$  is called a **scrambled set**.

The map  $f$  is called **distributionally chaotic** if  $\exists \epsilon > 0$  and an uncountable set  $\Gamma_\epsilon \subset X$  in which every pair of distinct points  $x, y$  satisfies

$$\liminf_{n \rightarrow \infty} \frac{1}{n} |\{k : d(f^k x, f^k y) < \epsilon, 0 \leq k < n\}| = 0$$

$$\text{and } \limsup_{n \rightarrow \infty} \frac{1}{n} |\{k : d(f^k x, f^k y) < \epsilon, 0 \leq k < n\}| = 1$$

and then  $\Gamma_\epsilon$  is called a **distributionally  $\epsilon$ -scrambled set**.

For example,

**every hypercyclic operator  $T$  on a Fréchet space  $F$  is Li-Yorke chaotic with respect to any (continuous) translation invariant metric: just fix a hypercyclic vector  $x$  and  $\Gamma = \{\lambda x : |\lambda| \leq 1\}$  is a scrambled set for  $T$ , [Martínez-Giménez et al. 2009].**



On Banach spaces, continuous Li-Yorke chaotic bounded linear operators  $T$  were characterized in terms of the existence of **irregular vectors**, where  $x$  is irregular for  $T$  if  $\liminf_n \|T^n x\| = 0$  and  $\limsup_n \|T^n x\| = \infty$ .

Sufficient ‘computable’ criteria for Li-Yorke chaos were given, and they established some additional conditions for the existence of dense scrambled sets. Further, **every infinite dimensional separable Banach space was shown to admit a distributionally chaotic operator which is also hypercyclic**, [Bermudez et al. 2011].

**However, there are examples of backward shifts on Köthe spaces of infinite-dimensional matrices which are uniformly distributionally chaotic and not hypercyclic.**

Köthe spaces provide a natural class of Fréchet sequence spaces in which many typical examples of weighted shifts are chaotic, [Martínez-Giménez et al. 2009].

**The existence of an uncountable scrambled set in Banach space setting need not be as strong an indicator of complicated dynamics as in the compact metric space case. It may happen that the span of a single vector becomes an uncountable scrambled set, [Bermudez et al. 2011].**

If an operator is hypercyclic, so it admits a vector with dense orbit, then it has a scrambled set in the strong sense of requiring linear independence of the vectors in the scrambled set, [Moothathu 2012].

Hypercyclic weighted translation operators and chaotic weighted translations on locally compact groups and their homogeneous spaces were characterized. The density of periodic points of a weighted translation implies hypercyclicity. However, a weighted translation operator is not hypercyclic if it is generated by a group element of finite order, [Chen, Chu 2009, Chen, Chu 2011, Chen 2011].


**A translation operator is never chaotic because its norm cannot exceed unity, but a weighted translation can be chaotic. It was known that for a unimodular complex number  $\alpha$  the rotation  $\alpha T$  of a hypercyclic operator on a complex Banach space is also hypercyclic but showed that on a separable Hilbert space there is a chaotic operator  $T$  with  $\alpha T$  not chaotic, [Bayart, Bermudez 2009].**







This is not the case for chaotic weighted translation operators because their rotations also are chaotic, [Chen 2011].















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