Harmonic-Killing vector fields on Kähler manifolds*

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Abstract

In a previous paper we have considered the harmonicity of local infinitesimal transformations associated to a vector field on a (pseudo)-Riemannian manifold to characterise intrinsically a class of vector fields that we have called harmonic-Killing vector fields. In this paper we extend this study to other properties, such as the pluriharmonicity and the α-pluriharmonicity (α harmonic 2-form) of the local infinitesimal transformations, obtaining characterisations of these kinds of vector fields.

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1 Introduction

In [1] we introduced the notion of a harmonic-Killing vector field, for which all of the 1-parameter groups of local transformations consist of harmonic maps. We introduced also the term 1-*harmonic-Killing* vector field for the case when the transformations have zero linear part of their tension field—which Nouhaud[8] had referred to as harmonic infinitesimal transformations. Such vector fields give rise to new examples of harmonic maps in pseudo-Riemannian geometry, and emphasise the importance of the complete lift metric for tangent bundles in the study of harmonicity.

With the objective of defining and characterising new types of vector fields, we consider other properties of the local linear transformations associated to a vector field on a (pseudo)-Riemannian manifold. First of all, in section 2 we review some of the results obtained in [1]. In section 3 we study harmonic-Killing vector fields in Kähler manifolds, obtaining that in the compact case such vector fields coincide with the holomorphic ones. We study next the vector fields for which 1-parameter groups of local transformations consist of pluriharmonic or α-pluriharmonic maps; we call such vector fields *pluriharmonic* or α-*pluriharmonic* vector fields, respectively. We end by obtaining intrinsic characterisations and giving relations among the new types of vector fields.

We begin by collecting some basic material that we need later.

2 Harmonic-Killing vector fields

Let (M, g) and (N, h) be Riemannian (or pseudo-Riemannian) manifolds with $\dim M = m$ and $\dim N = n$, and denote by ∇^M and ∇^N the Levi-Civita connections on M and N, respectively. A smooth map $\phi : (M, g) \to (N, h)$ defines a fibre bundle $\phi^*(TN)$, with projection $\pi_1 : \phi^*(TN) \to M$, $\pi_1(T_{\phi(x)}N) = x, \forall x \in M$. Sections of $\phi^*(TN)$, $\Gamma(\phi^*(TN))$, are called vector fields along ϕ. In particular, every tangent vector field on M, $X \in \Gamma(TM)$, induces a vector field $d\phi(X)$ along ϕ, such that $((d\phi(X))(x) = (d\phi)_x X(x), x \in M$. Moreover, every vector field X' on N, $X' \in \Gamma(TN)$, induces a vector field $X' \circ \phi$ along ϕ, such that $(X' \circ \phi)(x) = X'(\phi(x))$.

There exists a unique linear connection, $\phi^*\nabla^N$, on $\phi^*(TN)$ defined as follows

$$(\phi^*\nabla^N)_X(Y' \circ \phi)(x) = (\nabla^N_{d\phi(X)}Y') \circ \phi(x) = \nabla^N_{(d\phi)_x(X(x))}(Y'(\phi(x))),$$

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where $x \in M, X \in \Gamma(TM)$ and $Y' \in \Gamma(TN)$, $\phi^*\nabla^N$ is called the connection ϕ-induced by ∇^N on $\Gamma(\phi^*(TN))$.

This linear connection has the following properties:

(1) If ϕ is an immersion and $X, Y \in \Gamma(TM))$, then

$$(\phi^*\nabla^N)_X(d\phi(Y)) = \nabla^N_{d\phi(X)}(d\phi(Y)).$$

(2) As ∇^N is torsion free, we have for any $X, Y \in \Gamma(TM))$

$$(\phi^*\nabla^N)_X(d\phi(Y)) - (\phi^*\nabla^N)_Y(d\phi(X)) = d\phi([X, Y]).$$

(3) The metric h is parallel with respect to $\phi^*\nabla^N$, that is

$$(\phi^*\nabla^N)h = 0.$$

Let us denote by ∇' the naturally induced connection on the
tensor product $T^*M \otimes \phi^*(TM)$ by the connection ∇^M on T^*M, and the connection $\phi^*\nabla^N$ on $\phi^*(TN)$, then in particular

$$\big(\nabla'(d\phi)\big)_X(Y) = (\phi^*\nabla^N)_X\big(d\phi(Y)\big) - (d\phi)(\nabla^M_X Y)$$

is the second fundamental form of ϕ and the section $\tau(\phi) = trace_g\big(\nabla'(d\phi)\big)$ of $\phi^*(TN)$, is called the tension field of ϕ. Then, ϕ is said to be *harmonic* if $\tau(\phi) = 0$, and *totally geodesic* if $\big(\nabla'(d\phi)\big) = 0$. (See [2], [3] for more details and references.)

Now, let $U \subset M$ and $V \subset N$ be domains with coordinates $(x^1, \dots, x^m)$ and $(y^1, \dots, y^m)$ respectively, such that $\phi(U) \subset V$. Locally, the map ϕ has the representation: $y^a = \phi^a(x^1, \dots, x^m)$. Then the second fundamental form at $x \in U$ can be locally expressed by the following:

$$\big(\nabla'(d\phi)\big) = \big(\nabla'(d\phi)\big)^a_{ij} dx^i \otimes dx^j \otimes \big(\frac{\partial}{\partial y^a} \circ \phi\big)$$

where

$$\big(\nabla'(d\phi)\big)^a_{ij}(x) = \frac{\partial^2 \phi^a}{\partial x^i \partial x^j}(x) - {}^g\Gamma^k_{ij}(x)\frac{\partial \phi^a}{\partial x^k}(x) + {}^h\Gamma^a_{bc}(\phi(x))\big[\frac{\partial \phi^b}{\partial x^i}(x)\frac{\partial \phi^c}{\partial x^j}(x)\big], \tag{2.1}$$

for $i, j, k = 1, \dots, m$; $a, b, c = 1, \dots, n$.

If $\{U_i\}$ is an orthonormal reference for g in T_xM, then the tension field of ϕ has the following expression at $x \in U$

$$\tau(\phi)(x) = \sum_{i=1}^n \big(\nabla'(d\phi)\big)_{U_i}(U_i)(x). \tag{2.2}$$

With respect to the usual basis of T_xM $\{\frac{\partial}{\partial x^i}\}$ we have the expression

$$\begin{aligned}\tau(\phi)^a(x)\frac{\partial}{\partial y^a}(\phi(x)) &= g^{ij}(x)\big(\nabla'(d\phi)\big)^a_{\frac{\partial}{\partial x^i}}\big(\frac{\partial}{\partial x^j}\big)(x)\frac{\partial}{\partial y^a}(\phi(x)) \\ &= g^{ij}(x)\big(\nabla'(d\phi)\big)^a_{ij}(x)\frac{\partial}{\partial y^a}(\phi(x)),\end{aligned}$$

for $i, j = 1, \dots, m$; $a = 1, \dots, n$.

We note that slightly differing terminologies have arisen in the literature concerning the geometric characterization of vector fields through their groups of transformations. Nouhaud ([8, 9]) used the term harmonic infinitesimal transformations to mean that the local transformations have zero linear part of the tension field; we shall refer to this property as being 1-harmonic.

Definition 2.1 [1] *A vector field X on a pseudo-Riemannian m-manifold (M,g) is called a harmonic-Killing (1-harmonic-Killing) vector field if each local 1-parameter group of infinitesimal transformations associated to X is a group of harmonic (1-harmonic) maps. In this case we say that the infinitesimal transformation is a harmonic (1-harmonic) infinitesimal transformation.*

For harmonic-Killing vector fields, we have the following:

Theorem 2.1 [1] *Let (M,g) be a pseudo-Riemannian m-manifold and X a vector field on M. If X is a harmonic-Killing vector field then, for all $x \in M$, $\sum_j^m (\mathcal{L}_X \nabla)(Y_j, Y_j) = 0$, for any orthonormal frame, $\{Y_j\}, j = 1, \ldots, m$, on T_xM.* □

We have established the following equivalences for the definition of a 1-harmonic-Killing vector field.

Theorem 2.2 [1] *On a pseudo-Riemannian manifold (M,g) the following statements are equivalent:*

(i) *X is a 1-harmonic-Killing vector field.*

(ii) *$g^{ij}(\mathcal{L}_X \Gamma_{ij}^k) = 0,\ i,j,k = 1,\ldots,m$, where $\mathcal{L}$ denotes the Lie derivative and g^{ij} are the components of the inverse matrix of the metric g and Γ_{ij}^k are the Christoffel symbols of the Levi-Civita connection of g.*

(iii) *$X : (M,g) \longrightarrow (TM, g^C)$ is a harmonic section, where g^C denotes the complete lift of g.*

(iv) *$\triangle X = 2Ric(X,.)$, where $\triangle = d\delta + \delta d$, (d = differential, δ = codifferential) and Ric denotes the Ricci tensor of (M,g).*

(v) *X is a Jacobi field along the identity map of (M,g).*

□

Remark 2.1 Statement (v) is a particular case of Ferreira's Theorem [5], and (iii) is studied in [8]. (ii) ⇔ (iii) is proved in [9].

3 Harmonic-Killing vector fields in Kähler manifolds

Let M and N be complex manifolds with almost complex structures J_M and J_N, respectively. A C^∞-mapping $\phi : (M, J_M) \to (N, J_N)$ between two complex manifolds is holomorphic if the differential, $d\phi_x : T_xM \to T_{\phi(x)}N,\ x \in M$, satisfies:

$$J_N \circ d\phi_x = d\phi_x \circ J_M, \forall x \in M.$$

A Riemannian metric g on a complex manifold (M,J) is a Hermitian metric if $g(JX,JY) = g(X,Y),\ X,Y \in T_xM, \forall x \in M$,, Moreover if the 2-form ω given by $\omega(X,Y) := g(X,JY),\ X,Y \in T_xM$, $\forall x \in M$, is a closed form, i.e. $d\omega = 0$; then g is called a Kähler metric and (M,J,g) is called a Kähler manifold.

It is known that all holomorphic maps between Kähler manifolds are harmonic. We have also the following result for harmonic-Killing vector fields.

Proposition 3.1 *Let (M,J,g) be a compact Kähler manifold and X a vector field on M. Then X is harmonic-Killing if and only if X is holomorphic.*

Proof. We use the Lichnerowicz Rigidity Theorem ([2], p38 or [7]). If the transformations are harmonic variations of the identity, which is holomorphic, then they are holomorphic variations. □

In the case that (M,J,g) is not a compact manifold and for 1-harmonic-Killing the previous proposition does not work in general.

Example 3.1 Consider the complex Euclidean plane $\mathbb{C}$ with standard coordinates $z = x+iy$, and the Hermitian metric $g = dx \otimes dx + dy \otimes dy$. Let X be the vector field

$$X(x, y) = X^1(x, y)\frac{\partial}{\partial x} + X^2(x, y)\frac{\partial}{\partial y}.$$

The harmonic-Killing condition for X is

$$\frac{\partial^2 X^i}{\partial x \partial x} = -\frac{\partial^2 X^i}{\partial y \partial y}\ , \quad \text{with } i = 1, 2,$$

i.e., X^i, are harmonic functions from $\mathbb{R}^2$ to $\mathbb{R}$. Moreover the holomorphic condition for this type of example is the following

$$\frac{\partial X^1}{\partial x} = \frac{\partial X^2}{\partial y}, \quad \frac{\partial X^1}{\partial y} = -\frac{\partial X^2}{\partial x}.$$

Then $X^1 = \frac{1}{2}(x)^2 - \frac{1}{2}(y)^2$ and $X^2 = 0$ provide an X that is 1-harmonic-Killing but not holomorphic.

3.1 Pluriharmonic vector fields

Let (M, J, g) be a Kähler $2m$-manifold and (N, h) a Riemannian n-manifold. A C^∞-mapping $\phi : (M, J, g) \rightarrow (N, h)$ is called *pluriharmonic* if the second fundamental form of the map ϕ satisfies:

$$(\nabla d\phi)(X, Y) + (\nabla d\phi)(JX, JY) = 0, \quad X, Y \in T_x M, \ \forall x \in M.$$

Clearly a pluriharmonic map is a harmonic map. Also, it is well known that if N is a Kähler manifold and ϕ is a holomorphic map, then ϕ is pluriharmonic.

We define a new kind of vector field on Kähler manifolds.

Definition 3.1 *A vector field X on a Kähler 2m-manifold (M, J, g) is called a pluriharmonic (1-pluriharmonic) vector field if each local 1-parameter group of infinitesimal transformations associated to X, is a group of pluriharmonic (1-pluriharmonic) maps. In this case we say that the infinitesimal transformation is a pluriharmonic (1-pluriharmonic) infinitesimal transformation.*

It is well known that any vector field $X \in \Gamma(TM)$ gives rise to a local 1-parameter group of diffeomorphisms $I \ni t \mapsto \varphi_t \in \text{Diff}(M)$, where I is some neighborhood of $0 \in \mathbb{R}$, by solving the autonomous system of ordinary differential equations,

$$ev|_{t=t_0} \circ \frac{\partial}{\partial t} \circ \varphi_t^* = ev|_{t=t_0} \circ \varphi_t^* \circ X$$

with the equality understood as maps from $C^\infty(M)$ into itself, and subject to the initial condition $ev|_{t=0} \circ \varphi_t^* = \text{id}$, as an equality of algebra automorphisms of $C^\infty(M)$. This equation has a unique solution; namely, $\varphi_t^* = \exp(tX)$, where exp is defined through its Taylor series expansion, and X^k is understood as $X \circ \cdots \circ X$ (k-times). Our goal is to seek conditions under which φ_t is pluriharmonic for all $t \in I$. Now, the Lie algebra action of $\Gamma(TM)$ on the various geometrical objects defined on M is given through the following 'rule': Take the derivative with respect to t, evaluated at $t = 0$ of the Diff(M) action defined by the 1-parameter group of diffeomorphisms $\varphi_t \in$ Diff(M) associated to $X \in \Gamma(TM)$. Thus, the Lie algebra action of $\Gamma(TM)$ on $C^\infty(M)$ is given by

$$C^\infty(M) \times \Gamma(TM) \rightarrow C^\infty(M)\,, \quad (f, X) \mapsto \mathcal{L}_X f = X(f),$$

because $X(f) = ev|_{t=0} \circ \dfrac{\partial}{\partial t} \circ \varphi_t^* f$, and the right hand side is, in view of the differential equation, equal to $ev|_{t=0} \circ \varphi_t^* \circ X\,(f) = X(f)$.

Similarly, the Lie algebra action of $\Gamma(TM)$ on connections on M, $(Con(M))$, is given by

$$Con(M) \times \Gamma(TM) \to Con(M)\,, \quad (\nabla, X) \mapsto \mathcal{L}_X \nabla\,.$$

Namely,

$$\mathcal{L}_X \nabla = ev|_{t=0} \circ \frac{\partial}{\partial t} \circ \nabla^{\varphi_t}\,,$$

where ∇^{φ_t} is the result of the natural action of $\Gamma(TM)$ on $Con(M)$, that is,

$$\nabla^{\varphi_t}_Z Y = \varphi^*_t \circ (\nabla_{Z^{\varphi_{-t}}} Y^{\varphi_{-t}}) \circ \varphi^*_{-t},$$

and by W^φ we denote the right action of Diff(M) on $\Gamma(TM)$, i.e., $W^\varphi = \varphi^* \circ W \circ \varphi^{-1*}$, $\varphi \in$Diff(M), $W \in \Gamma(TM)$.

Now we are ready to give an intrinsic characterisation for a 1-pluriharmonic vector field.

Theorem 3.1 *Let (M, J, g) be a Kähler $2m$-manifold and X a vector field on M. The vector field X is a 1-pluriharmonic vector field if and only if $(\mathcal{L}_X\nabla)(Y,Z) + (\mathcal{L}_X\nabla)(JY, JZ) = 0$, $\forall Y, Z$.*

Proof. If X is a vector field on M the differential $d\varphi_t$ defines a section of the vector bundle $T^*M \otimes \varphi^*_t(TM) \simeq Hom(TM, \varphi^*_t TM)$, where $\varphi^*_t(TM)$ is the pullback bundle of TM along φ_t. In fact,

$$\Gamma(TM) \ni Z \mapsto d\varphi_t(Z) = Z \circ \varphi^*_t \in \Gamma(\varphi^*_t TM).$$

Indeed,

$$Z \circ \varphi^*_t = \varphi^*_t \circ \varphi^*_{-t} \circ Z \circ \varphi^*_t = \varphi^*_t \circ Z^{\varphi_{-t}}.$$

Let us denote by ∇' the naturally induced connection on the tensor product $T^*M \otimes \varphi^*_t(TM)$ by ∇ the connection on T^*M, and by $\varphi^*_t\nabla$ the connection on $\varphi^*_t(TM)$, then

$$\big(\nabla'(d\varphi_t)\big)_Z(Y) = (\varphi^*_t\nabla)_Z\big(d\varphi_t(Y)\big) - (d\varphi_t)(\nabla_Z Y) = \varphi^*_t \circ \nabla_{Z^{\varphi_{-t}}} Y^{\varphi_{-t}} - (d\varphi_t)(\nabla_Z Y).$$

Note, in particular, that the assignment $(Z, Y) \mapsto \big(\nabla'(d\varphi_t)\big)_Z\big(Y\big)$ is symmetric when the connection on TM is the Levi-Civita connection, since the difference $\big(\nabla'(d\varphi_t)\big)_Z(Y) - \big(\nabla'(d\varphi_t)\big)_Y(Z)$ equals $\varphi^*_t([Z^{\varphi_{-t}}, Y^{\varphi_{-t}}]) - (d\varphi_t)([Z,Y])$, which vanishes identically.

To say that φ_t is pluriharmonic, is to say that

$$\big(\nabla'(d\varphi_t)\big)_{Y_i}(Y_j) \circ \varphi_t^{-1} + \big(\nabla'(d\varphi_t)\big)_{JY_i}(JY_j) \circ \varphi_t^{-1} = 0,\ \forall i, j = 1, \ldots, m, \tag{3.1}$$

where $\{Y_k, JY_k\}$, $k = 1, \ldots, m$, is a frame on TM. So, after substituing the expression of $\nabla'(d\varphi_t)$ we get that (3.1) is equivalent to

$$\nabla_{Y_i{}^{\varphi_{-t}}} Y_j^{\varphi_{-t}} - (\nabla_{Y_i} Y_j)^{\varphi_{-t}} + \nabla_{JY_i{}^{\varphi_{-t}}} JY_j^{\varphi_{-t}} - (\nabla_{JY_i} JY_j)^{\varphi_{-t}} = 0,\ \forall i, j = 1, \ldots, m.$$

The corresponding infinitesimal condition is therefore that,

$$ev|_{t=0} \frac{\partial}{\partial t} \circ (\nabla_{Y_i{}^{\varphi_{-t}}} Y_j^{\varphi_{-t}} - (\nabla_{Y_i} Y_j)^{\varphi_{-t}} + \nabla_{JY_i{}^{\varphi_{-t}}} JY_j^{\varphi_{-t}} - (\nabla_{JY_i} JY_j)^{\varphi_{-t}}) = 0.$$

Computing the derivatives on the left hand side, and simplifying, we obtain

$$\mathcal{L}_X\nabla(Y_i, Y_j) + \mathcal{L}_X\nabla(JY_i, JY_j) = 0,\ \forall i, j = 1, \ldots, m,$$

where X is the vector field whose 1-parameter group of diffeomorphisms is φ_t, and this gives the result. In fact, we have proved that X being pluriharmonic implies the condition in the theorem, but we need 1-pluriharmonic for the converse, as may be seen from the general result for second fundamental form in Nouhaud[8]. □

We obtain also the following equivalence.

Theorem 3.2 *A vector field X on a Kähler $2m$-manifold (M,J,g) is a 1-pluriharmonic vector field if and only if the section $X:(M,J,g)\longrightarrow(TM,g^C)$ is a pluriharmonic map, where g^C denotes the complete lift of g.*

Proof. If we consider the vector field $X=X^a\frac{\partial}{\partial x^a}$, $a=1,\ldots,2m$, as a map from (M,J,g) to (TM,g^C) (see [12] for the description of the complete lift g^C) we have in coordinates:

$$X:(M,g)\longrightarrow(TM,g^C),\ x^a\mapsto X(x^a)=(x^a,X^a),\ a=1,\ldots,2m.$$

The local expression of the second fundamental form of X is the following (see [8]):

$$(\nabla dX)^c(\frac{\partial}{\partial x^a},\frac{\partial}{\partial x^b})=0$$

$$(\nabla dX)^{\bar{c}}(\frac{\partial}{\partial x^a},\frac{\partial}{\partial x^b})=(\mathcal{L}_X\nabla)^c(\frac{\partial}{\partial x^a},\frac{\partial}{\partial x^b})$$

where $a,b,c=1,\ldots,2m$, and $\bar{c}=c+2m$.

If we apply this formula to the basis $\{\frac{\partial}{\partial x^i},J(\frac{\partial}{\partial x^i})\}$, $i=1,\ldots,m$, the result follows from the linearity of the Lie derivative. □

It is well known that a pluriharmonic map is harmonic and holomorphic maps are pluriharmonic, moreover, in the case of pluriharmonic vector fields on compact Kähler manifolds we have the following equivalences.

Proposition 3.2 *Let (M,J,g) be a compact, Kähler manifold and X a vector field on M. Then the following conditions are equivalent:*

(i) *X is holomorphic,*

(ii) *X is harmonic-Killing,*

(iii) *X is pluriharmonic.*

Proof. We have that a holomorphic map $\phi:M\longrightarrow N$ is a pluriharmonic map if N is a Kähler manifol. Then, following the Proposition 3.1, we have the equivalences. □

3.2 α-Pluriharmonic vector fields

We have that a vector field X on a Kähler manifold (M,J,g) is a pluriharmonic vector field if the J-invariant part of its second fundamental form vanishes. Our goal is to generalize this concept when M is not necessarily a Kähler manifold. Following this objective we introduce the Clifford formalism (see [6] for a complete description).

Let (M,g) and (N,g) be Riemannian (or pseudo-Riemannian) manifolds, connected, without boundary and with $\dim M=m\geq 2$ and $\dim N=n\geq 2$. Let $\phi:(M,g)\longrightarrow(N,h)$ be a smooth map. We denote by $D=d+\delta$ the Dirac operator of the Dirac bundle of M, acting on differential forms with values on

$\phi^*(TN)$. If α is a p-form on M and σ is a section of $\phi^*(TN)$, the Clifford multiplication, $*$, is defined by

$$\sigma*\alpha=\sigma\wedge\alpha-\iota(\sigma)\alpha,\ \text{and}\ \alpha*\sigma=(-1)^p(\sigma\wedge\alpha+\iota(\sigma)\alpha),$$

where $\wedge$ and ι denote the exterior product and the interior product of differential forms, respectively.

If $\{e_i\}_{i=1}^{i=m}$ is an orthonormal reference then D has the following expression:

$$D=\sum_i e_i*\nabla e_i.$$

So, the smooth map $\phi:(M,g)\longrightarrow(N,h)$ is said to be *α-pluriharmonic*, where α is a harmonic 2-form ($d\alpha=\delta\alpha=0$) on M, when

$$D(\alpha * d\phi) - \alpha * D(d\phi) = 0.$$

Following the structure of the previous sections we introduce the α-pluriharmonic vector fields characterised by the property that their integral flows act on the manifold by means of α-pluriharmonic diffeomorphisms.

Definition 3.2 *Let (M, g) be a pseudo-Riemannian manifold. A vector field X on M is called a α-pluriharmonic (1-α-pluriharmonic) vector field if each local 1-parameter group of infinitesimal transformations associate to X is a group of α-pluriharmonic (1-α-pluriharmonic) maps. In this case we say that the infinitesimal transformation is an α-pluriharmonic (a 1-α-pluriharmonic) infinitesimal transformation.*

Lemma 3.1 [10] *If α is a harmonic 2-form over M, then for all $x \in M$, we have the following:*

$$D(\alpha * d\phi) - \alpha * D(d\phi) \quad = \quad 2\sum_i \nabla d\phi(e_i, .) \wedge \alpha(e_i, .),$$

where $*$ denote the Clifford multiplication and $\{e_i\}_{i=1}^{i=m}$ is an orthonormal local frame on T_xM.

Theorem 3.3 *Let (M, g) be a Riemannian (or pseudo-Riemannian) manifold. A vector field X on M is a 1-α-pluriharmonic vector field if and only if, for all $x \in M$,*

$$\sum_i^m (\mathcal{L}_X \nabla)(e_i, .) \wedge \alpha(e_i, .) = 0,$$

for any orthonormal local frame $\{e_i\}_{i=1}^{i=m}$ on T_xM.

Proof. We follows the notation of the previous sections. If X is a vector field on M with infinitesimal transformation φ_t, then to say that φ_t is α-pluriharmonic, following the Lemma 3.1, is to say that, for any $x \in M$,

$$\sum_i^m (\nabla' d\varphi_t(e_i, .) \circ \varphi_{-t} \wedge \alpha(e_i, .) \quad = \quad 0, \tag{3.2}$$

and (3.2) vanishes if and only if for any orthonormal frame $\{e_i\}_{i=1}^{i=m}$ on T_xM,

$$\sum_i^m [(\alpha(e_i, e_k) \nabla' d\varphi_t(e_i, e_j) \circ \varphi_{-t}) - (\alpha(e_i, e_j) \nabla' d\varphi_t(e_i, e_k) \circ \varphi_{-t})] = 0, \ \forall j, k = 1, \ldots, m.$$

So, after substituing the expression of $\nabla' d\varphi_t$ we get that (3.2) is equivalent to

$$\sum_i^m \{\alpha(e_i, e_k)((\nabla_{e_i^{\varphi_t}} e_j^{\varphi_t}) - (\nabla_{e_i} e_j)^{\varphi^{-t}})) - \alpha(e_i, e_j)((\nabla_{e_i^{\varphi_t}} e_k^{\varphi_t}) - (\nabla_{e_i} e_k)^{\varphi^{-t}}))\} = 0, \forall j, k = 1, \ldots, m.$$

The corresponding infinitesimal condition is therefore that

$$ev|_{t=0} \frac{\partial}{\partial t} \circ \{\sum_i^m [\alpha(e_i, e_k)((\nabla_{e_i^{\varphi_t}} e_j^{\varphi_t}) - (\nabla_{e_i} e_j)^{\varphi^{-t}})) - \alpha(e_i, e_j)((\nabla_{e_i^{\varphi_t}} e_k^{\varphi_t}) - (\nabla_{e_i} e_k)^{\varphi^{-t}}))]\} = 0,$$

$\forall j, k = 1, \ldots, m.$
Computing the derivatives on the left hand side, and simplifying, we obtain

$$\sum_{i=1}^m \alpha(e_i, e_k)(\mathcal{L}_X \nabla)(e_i, e_j) - \alpha(e_i, e_j)(\mathcal{L}_X \nabla)(e_i, e_k) = 0, \ \forall j, k = 1, \ldots, m.$$

The condition is, therefore,

$$\sum_i^m (\mathcal{L}_X \nabla)(e_i, .) \wedge \alpha(e_i, .) = 0,$$

where X is the vector field whose 1-parameter group of diffeomorphisms is φ_t, and this proves the result. Again, we have proved slightly more than the statement, since $\alpha-$pluriharmonic implies the condition, but here again we need 1-αpluriharmonic for the converse by Nouhaud[8]. □

Theorem 3.4 *Let (M, g) be a Riemannian (or pseudo-Riemannian) manifold. A vector field X is a 1-α-pluriharmonic vector field if and only if the section $X : (M, g) \longrightarrow (TM, g^C)$ is an α-pluriharmonic map, where g^C denotes the complete lift of g.*

Proof. As in Theorem 3.2, if we consider the vector field $X = X^i \frac{\partial}{\partial x^i}$, $i = 1, \ldots, m$, as a map from (M, g) to (TM, g^C), the local expression of the second fundamental form of X, at the point $x \in M$ is the following:

$$(\nabla dX)^k(e_i, e_j) = 0$$

$$(\nabla dX)^{\bar{k}}(e_i, e_j) = (\mathcal{L}_X \nabla)^k(e_i, e_j),$$

where $\{e_i\}_{i=1}^{i=m}$ is an orthonormal frame of T_xM, $i, j, k = 1 \ldots, m$ and $\bar{k} = k + m$. This expression and Theorem 3.3 proves the result. □

It is well know that an α-pluriharmonic map, with α non-degenerate, is a harmonic map. Moreover, in the case of pluriharmonic or α-pluriharmonic vector fields we have the following result.

Proposition 3.3 *Let be (M, J, g) a compact, Kähler manifold and X a vector field on M. Then the following conditions are equivalent:*

(i) *X is holomorphic,*

(ii) *X is harmonic-Killing,*

(iii) *X is pluriharmonic,*

(iv) *X is ω-pluriharmonic, where ω is the Kähler 2-form over M.*

Proof. The first three equivalences are the Proposition 3.2. Moreover, making easy calculations we obtain that

$$(D(\omega * d\varphi_t) - \omega * (d\varphi_t))(X, Y) \quad = \quad 2((\nabla d\varphi_t)(JX, Y) - (\nabla d\varphi_t)(X, JY)),$$

where φ_t is the 1-parameter group of transformations associated to X. So, we have the equivalence $(iii) \Leftrightarrow (iv)$, and this completes the proof. □

In the presence of conditions on the curvature of the manifold, there exist relations among these concepts. If (M, g) is a compact Riemannian manifold with non-positive complex sectional curvature and α is a parallel 2-form on M, then, using [4] (Proposition 3.1), a vector field X on M is harmonic-Killing if and only if it is α-pluriharmonic.

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