

A GENERALIZED SECOND ORDER FRAME BUNDLE FOR FRÉCHET MANIFOLDS

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ABSTRACT. Working within the framework of Fréchet modelled infinite dimensional manifolds, we propose a generalized notion of second order frame bundle. We revise in this way the classical notion of bundles of linear frames of order two, the direct definition and study of which is problematic due to intrinsic difficulties of the space models. However, this new structure keeps all the fundamental characteristics of a frame bundle: It is a principal Fréchet bundle associated (differentially and geometrically) with the corresponding second order tangent bundle.

INTRODUCTION

In a previous paper ([6]) we studied the second order tangent bundle T^2M of a smooth infinite dimensional manifold M , i.e. the bundle of curves of M that agree up to their acceleration. More precisely, a vector bundle structure developed on T^2M provided that M is endowed with a linear connection. This study embraced the case of Banach modelled manifolds as well as that of a certain type of Fréchet manifolds.

Motivated by the fact that a great number of geometric properties of a vector bundle can be reduced to the corresponding bundle of linear frames, we present in this note a detailed study of second order frame bundles.

The case of a Banach modelled manifold M , has already been covered in [7], where we gave a rather natural extension of the results obtained by Dodson and Radioivici in [8], concerning the classical case of finite dimensional manifolds. For M modelled on a Banach space \mathbb{E} , the frame bundle of order two becomes

$$L^2M = \bigcup_{x \in M} \mathcal{L}is(\mathbb{E} \times \mathbb{E}, T_x^2M).$$

This is a Banach principal bundle with structure group $GL(\mathbb{E} \times \mathbb{E})$ associated with the second order tangent bundle. Moreover, a bijective correspondence between connections on T^2M and L^2M can be established.

However, such an approach does not extend to the case of an infinite dimensional manifold M modelled on a Fréchet space \mathbb{F} . In this framework, the general linear group $GL(\mathbb{F} \times \mathbb{F})$ does not admit any reasonable Lie group structure and, therefore, even the definition of the second order frame bundle is under question. On the other hand, intrinsic difficulties of the spaces of linear mappings between Fréchet spaces as well as the lack of a general solvability theory of differential equations, set serious obstacles in the study of the corresponding geometric entities.

Here we focus on a wide class of Fréchet manifolds: those which can be obtained as projective limits of Banach manifolds (several examples of these types of structures, which seem to be popular among theoretical physicists, can be found in [1], [10], [11]). Within this framework we bypass the aforementioned difficulties by

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introducing a generalized bundle of frames, which has no longer as structure group the pathological general linear group of the fibre type, but a new topological, and in a generalized sense smooth, Lie group. The latter can be also thought of as a subgroup of $GL(\mathbb{F} \times \mathbb{F})$ since it consists of projective systems of linear isomorphisms (cf. Preliminaries Section).

The tangent bundle T^2M proves then to be associated with this generalized bundle of frames (Theorem 2.5) while, at the same time, a one to one correspondence between their connections is revealed (Theorem 3.1).

The paper is concluded with some suggestions of areas of application in physics.

1. PRELIMINARIES

In this first section we present all the basic preliminary notions needed in the sequel of the paper. We begin with a short description of the type of infinite dimensional manifolds that we are going to use.

Let $\{M^i; \varphi^{ji}\}_{i,j \in \mathbb{N}}$ be a projective (inverse) system of Banach manifolds modelled on the Banach spaces $\{\mathbb{F}^i\}_{i \in \mathbb{N}}$ respectively. This means that the mappings $\varphi^{ji} : M^j \rightarrow M^i$ ($j \geq i$) are smooth and satisfy the following conditions for every choice of indices (i, j, k) with $j \geq i \geq k$:

$$\varphi^{ik} \circ \varphi^{ji} = \varphi^{jk}.$$

We further demand that the models \mathbb{F}^i form a projective system with connecting morphisms $\rho^{ji} : \mathbb{F}^j \rightarrow \mathbb{F}^i$ ($j \geq i$), and limit $\mathbb{F} = \varprojlim \mathbb{F}^i$. Assuming that for all $x = (x^i) \in M := \varprojlim M^i$ there exists a projective system of local charts $\{(U^i, \psi^i)\}_{i \in \mathbb{N}}$ such that $x^i \in U^i$ and the corresponding limits $\varprojlim U^i$, $\varprojlim \psi^i(U^i)$ are open, $M = \varprojlim M^i$ turns out to be a Fréchet manifold modelled on \mathbb{F} . The corresponding local structure is defined by the charts $\{(\varprojlim U^i, \varprojlim \psi^i)\}$. Moreover, the tangent bundle TM of M also can be endowed with a Fréchet manifold structure with model the Fréchet space $\mathbb{F} \times \mathbb{F}$ and local trivializations defined by the projective limits of the differentials $\{T\psi^i\}$ being, thus, isomorphic to $\varprojlim TM^i$.

Concerning the differentiability of mappings between Fréchet spaces we adopt the definition of Leslie ([15], [16]). However, the differentiability proposed by Kriegl and Michor ([14]) also suits our study.

Some of the main difficulties that one faces in the study of Fréchet manifolds are direct reflections of intrinsic problems of their models. As already discussed in the Introduction, these drawbacks are mainly related with the space of continuous linear mappings. Indeed, within the framework of Fréchet spaces the latter do not remain in the same category, although they are endowed with a topological vector space structure. On the other hand, the corresponding general linear groups fail to be smooth Lie groups or even (non trivial) topological groups.

A partial way out, at least, is given by the replacement of the abovementioned pathological structures by a new construction that allows us to work successfully within the framework of non Banach spaces: If \mathbb{F}, \mathbb{G} are two Fréchet spaces, we take advantage of the fact that *always* they can be realized as a projective limit of Banach spaces $\mathbb{F} = \varprojlim \mathbb{F}^i$, $\mathbb{G} = \varprojlim \mathbb{G}^i$ (see e.g. [23]), and we define

$$H(\mathbb{F}, \mathbb{G}) := \{(l^i)_{i \in \mathbb{N}} \in \prod_{i=1}^{\infty} \mathcal{L}(\mathbb{F}^i, \mathbb{G}^i) : \varprojlim l^i \text{ exists}\}.$$

This is a Fréchet space (under the Cartesian product topology) and is going to replace in our study the space of continuous linear mappings $\mathcal{L}(\mathbb{F}, \mathbb{G})$. Focusing now on the invertible elements of the above structure, we may also define the

topological group

$$H_0(\mathbb{F}, \mathbb{G}) := \{(l^i)_{i \in \mathbb{N}} \in \prod_{i=1}^{\infty} \mathcal{L}is(\mathbb{F}^i, \mathbb{G}^i) : \varprojlim l^i \text{ exists}\},$$

which is isomorphic to the projective limit of the Banach Lie groups

$$H_0^i(\mathbb{F}, \mathbb{G}) := \{(l^1, l^2, \dots, l^i)_{i \in \mathbb{N}} \in \prod_{k=1}^i \mathcal{L}is(\mathbb{F}^k, \mathbb{G}^k) : \rho^{mk} \circ l^m = l^k \circ \rho^{mk}, i \geq m \geq k\}.$$

In view of the previous definitions, we plan to replace the pathological general linear group $GL(\mathbb{F})$ by

$$H_0(\mathbb{F}) := H_0(\mathbb{F}, \mathbb{F}) = \{(l^i)_{i \in \mathbb{N}} \in \prod_{i=1}^{\infty} GL(\mathbb{F}^i) : \varprojlim l^i \text{ exists}\}.$$

The latter can be thought of also as a generalized Fréchet Lie group by being embedded in $H(\mathbb{F}) := H(\mathbb{F}, \mathbb{F})$.

Using the above methodology, we have defined in [6] a Fréchet vector bundle structure on the second order tangent bundle (T^2M, π_2, M) , which consists of all equivalence classes of curves in M that agree up to their acceleration. To be more specific, if $\{(U_\alpha = \varprojlim U_\alpha^i, \psi_\alpha = \varprojlim \psi_\alpha^i)\}_{\alpha \in I}$ is an atlas of M and $D : T(TM) \rightarrow TM$ a linear connection, obtained as a projective limit of corresponding connections on the factors $D^i : T(TM^i) \rightarrow TM^i$ ($i \in \mathbb{N}$) and associated to a family of Christoffel symbols $\{\Gamma_\alpha = \varprojlim \Gamma_\alpha^i : \mathbb{F} \rightarrow H(\mathbb{F}, H(\mathbb{F}, \mathbb{F}))\}_{\alpha \in I}$ (see [12] for a detailed study of tangent bundles and connections obtained as projective limits), then the following local structure can be defined on T^2M :

$$\Phi_\alpha : \pi_2^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{F} \times \mathbb{F}$$

with

$$(1) \quad \Phi_\alpha([f, x]_2) = (x, (\psi_\alpha \circ f)'(0), (\psi_\alpha \circ f)''(0) + \Gamma_\alpha(\psi_\alpha(x))((\psi_\alpha \circ f)'(0))[(\psi_\alpha \circ f)'(0)]),$$

where $[f, x]_2$ stands for the equivalence class of the smooth curve $f : \mathbb{R} \rightarrow M$ with respect to the relation

$$f \approx_x g \Leftrightarrow f(0) = g(0) = x, f'(0) = g'(0) \text{ and } f''(0) = g''(0).$$

Under these trivializations, T^2M turns out to be a vector bundle with fibre type $\mathbb{F} \times \mathbb{F}$ and structure group $H_0(\mathbb{F} \times \mathbb{F})$. Moreover, T^2M can be thought of as a projective limit of Banach vector bundles since it coincides with the limit of the projective system $\{T^2M^i; g^{ji}\}_{i, j \in \mathbb{N}}$, where the connecting morphisms g^{ji} are defined by

$$g^{ji} : T^2M^j \longrightarrow T^2M^i : [f, x]_2^j \mapsto [\phi^{ji} \circ f, \phi^{ji}(x)]_2^i,$$

where $[f, x]_2^j$ denotes the equivalence class of all curves in M^j that agree up to their acceleration with f .

2. SECOND ORDER FRAME BUNDLES OF FRÉCHET MANIFOLDS

Having established in the previous section all the necessary preliminary material, we present here the main result of this paper: The construction of a frame bundle of order two on a Fréchet modelled manifold.

In previous work of Dodson and Radivoioci ([8]) the case of a finite dimensional manifold M modelled on Euclidean space \mathbb{R}^n has been developed successfully. More

precisely, the frame bundle of order two L^2M of M is defined, classically, to be the union

$$\bigcup_{x \in M} \mathcal{L}is(\mathbb{R}^n \times \mathbb{R}^n, T_x^2M),$$

where T_x^2M is the second order tangent space of M over x . Then, L^2M proves to be a principal fibre bundle over M , with structure group the general linear group $GL(\mathbb{R}^n \times \mathbb{R}^n)$, associated with the second order tangent bundle T^2M . This construction works well also within the framework of Banach manifolds as detailed in [7].

However, any attempt to apply this methodology to a manifold M modelled on a Fréchet space \mathbb{F} is doomed to failure in consequence of intrinsic problems with the space model. The main difficulty is with the general linear group $GL(\mathbb{F} \times \mathbb{F})$ of the fibre type which is almost useless here since it does not admit any reasonable Lie (or even topological) group structure. As a result, the definition of a smooth fibre bundle structure on L^2M , along the above lines, is not possible.

Our aim here is to overcome these problems by defining a generalized notion of second order frame bundle for a wide class of Fréchet manifolds: Those that can be obtained as projective limits of Banach manifolds.

To this end, let $M = \varprojlim M^i$ be such a manifold, as explicitly defined in the Preliminaries Section, with connecting morphisms $\{\varphi^{ji} : M^j \rightarrow M^i\}_{i,j \in \mathbb{N}}$ and space model the limit \mathbb{F} of a projective system of Banach spaces $\{\mathbb{F}^i; \rho^{ji}\}_{i,j \in \mathbb{N}}$. Following the results obtained in [6], if we assume further that M is endowed with a linear connection $D = \varprojlim D^i$, then T^2M admits a vector bundle structure over M with fibres of Fréchet type $\mathbb{F} \times \mathbb{F}$. More precisely, T^2M becomes also a projective limit of manifolds via the identification $T^2M \simeq \varprojlim T^2M^i$. The corresponding local trivializations are those of relation (1) in the Preliminaries Section.

Our strategy from here can be briefly described as follows:

- We define a principal Banach fibre bundle over each factor manifold M^i ($i \in \mathbb{N}$) that generalizes the classical frame bundle of order two.
- We prove that these bundles form a projective system with limit the desired generalization of the second order frame bundle of the Fréchet manifold M .

To this end, for each $i \in \mathbb{N}$, we define

$$\begin{aligned} \mathcal{F}^2M^i &= \bigcup_{x^i \in M^i} \{(h^k)_{k=1, \dots, i} : h^k \in \mathcal{L}is(\mathbb{F}^k \times \mathbb{F}^k, T_{\varphi^{ik}(x^i)}^2M^k)\} \text{ and} \\ g^{mk} \circ h^m &= h^k \circ (\rho^{mk} \times \rho^{mk}), \quad i \geq m \geq k. \end{aligned}$$

Then, basic for the sequel, the next result can be proved.

Proposition 2.1. \mathcal{F}^2M^i is a principal fibre bundle over M^i with structure group the Banach Lie group $H_0^i(\mathbb{F} \times \mathbb{F}) := H_0^i(\mathbb{F} \times \mathbb{F}, \mathbb{F} \times \mathbb{F})$.

Proof. In view of the realization of a Fréchet space as a projective limit of Banach spaces (see, e.g., [23]), we may assume that the canonical projections $\rho^i : \mathbb{F} \rightarrow \mathbb{F}^i$, $\varphi^i : M \rightarrow M^i$ ($i \in \mathbb{N}$) are surjective. As a result, for all $x^i \in M^i$ there exists an element $x \in M$ with $\varphi^i(x) = x^i$. Choosing a chart $(U_\alpha = \varprojlim U_\alpha^i, \psi_\alpha = \varprojlim \psi_\alpha^i)$, $a \in I$, of M through x , we construct the corresponding trivialization

$$(\pi_2^{-1}(U_\alpha) = \varprojlim (\pi_2^i)^{-1}(U_\alpha^i), \Phi_\alpha = \varprojlim \Phi_\alpha^i)$$

of T^2M (compare with relation (1)) as well as the linear isomorphisms

$$\tau_\alpha := pr_2 \circ \Phi_\alpha|_{\pi_2^{-1}(x)} = \varprojlim \tau_\alpha^i = \varprojlim (pr_2 \circ \Phi_\alpha^i|_{(\pi_2^i)^{-1}(x^i)}),$$

where pr_2 denotes the projection of $U_\alpha \times \mathbb{F}^2$ to the second factor.

Define the map

$$p^i : \mathcal{F}^2 M^i \rightarrow M^i : (h^1, h^2, \dots, h^i) \mapsto x^i,$$

for all (h^1, h^2, \dots, h^i) with $h^i \in \mathcal{L}is(\mathbb{F}^i \times \mathbb{F}^i, T_{x^i}^2 M^i)$ and the action

$$(2) \quad (h^1, h^2, \dots, h^i) \cdot (g^1, g^2, \dots, g^i) := (h^1 \circ g^1, h^2 \circ g^2, \dots, h^i \circ g^i)$$

of $H_0^i(\mathbb{F} \times \mathbb{F})$ on the right of $\mathcal{F}^2(M^i)$; then we may check that the mappings

$$\Psi_\alpha^i : (p^i)^{-1}(U_\alpha^i) \rightarrow U_\alpha^i \times H_0^i(\mathbb{F} \times \mathbb{F}) : (h^1, \dots, h^i) \mapsto (p^i(h^1, \dots, h^i), \tau_\alpha^1 \circ h^1, \dots, \tau_\alpha^i \circ h^i)$$

are well defined bijections. Indeed, the injectivity is a direct consequence of the fact that each τ_α^i is a linear isomorphism and the surjectivity is due to the realization of every element $(x^i, g^1, \dots, g^i) \in U_\alpha^i \times H_0^i(\mathbb{F} \times \mathbb{F})$ in the form $\Psi_\alpha^i((\tau_\alpha^1)^{-1} \circ g^1, \dots, (\tau_\alpha^i)^{-1} \circ g^i)$.

As a result, for each $a \in I$, the set

$$X_a := (p^i)^{-1}(U_\alpha^i)$$

can be endowed with a smooth manifold structure via Ψ_α^i modelled on the Banach space $\mathbb{F}^i \times H_0^i(\mathbb{F} \times \mathbb{F})$. Noticing that, for every pair of indices $(a, b) \in I^2$, the intersection $\Psi_\alpha^i(X_a \cap X_b) = (U_\alpha^i \cap U_\alpha^b) \times H_0^i(\mathbb{F} \times \mathbb{F})$ is open in $\Psi_\alpha^i(X_a)$, we may conclude that $X_a \cap X_b$ is open in X_a .

On the other hand, we may verify that the differential structure of $X_a \cap X_b$ as a submanifold of X_a coincides with the one bequeathed by X_b . For this, it suffices to prove that the mapping $\Psi_b^i \circ (\Psi_\alpha^i)^{-1}$ is a diffeomorphism of $(U_\alpha^i \cap U_\alpha^b) \times H_0^i(\mathbb{F} \times \mathbb{F})$. Indeed, the corresponding definitions lead to

$$(\Psi_b^i \circ (\Psi_\alpha^i)^{-1})(x, g^1, \dots, g^i) = (x, ((comp \circ (T_{\alpha b}^k \times pr_k))(x, g^1, \dots, g^i))_{k=1,2,\dots,i}),$$

where *comp* stands for the composition mapping

$$\mathcal{L}(\mathbb{F}^k) \times \mathcal{L}(\mathbb{F}^k) \rightarrow \mathcal{L}(\mathbb{F}^k) : (f, g) \mapsto f \circ g,$$

$(T_{\alpha b}^k)_{a,b \in I}$ are the transition functions of the second order tangent bundle $T^2 M^k$ and $pr_k : \prod_{k=1}^i \mathcal{L}(\mathbb{F}^k) \rightarrow \mathcal{L}(\mathbb{F}^k)$ the projection to the k -th factor.

Based on this analysis, we may apply the well known gluing Lemma ([2]; N° 5.2.4) to conclude that $\mathcal{F}^2(M^i) = \bigcup_{a \in I} X_a$ is a principal Banach bundle as in the statement. \square

The previous construction allows us now to prove that the bundles $(\mathcal{F}^2 M^i)_{i \in \mathbb{N}}$ form a projective system. Indeed, considering, for every pair of indices $(i, j) \in \mathbb{N}^2$ with $j \geq i$, the projections

$$r^{ji} : \mathcal{F}^2 M^j \longrightarrow \mathcal{F}^2 M^i : (h^1, h^2, \dots, h^j) \mapsto (h^1, h^2, \dots, h^i),$$

as well as the corresponding ones on the structure groups

$$rh^{ji} : H_0^j(\mathbb{F} \times \mathbb{F}) \longrightarrow H_0^i(\mathbb{F} \times \mathbb{F}) : (g^1, g^2, \dots, g^j) \mapsto (g^1, g^2, \dots, g^i),$$

we easily verify that

$$r^{ik} \circ r^{ji} = r^{jk}, \quad rh^{ik} \circ rh^{ji} = rh^{jk} \quad (j \geq i \geq k).$$

As a result, the limit $\varprojlim \mathcal{F}^2 M^i$ exists and can be endowed with a principal bundle structure as illustrated in the next main result.

Theorem 2.2. *The limit $\varprojlim \mathcal{F}^2 M^i$ is a Fréchet principal bundle over M with structure group $H_0(\mathbb{F} \times \mathbb{F})$.*

Proof. We observe first that the trivializations of $\mathcal{F}^2 M^i$, $i \in \mathbb{N}$, defined in the previous Proposition, form projective systems since

$$(\phi^{j^i} \times r^{h^{j^i}}) \circ \Psi_\alpha^j = \Psi_\alpha^i \circ r^{j^i}, \quad (j \geq i).$$

Therefore, taking into account that $\varprojlim(H_0^i(\mathbb{F} \times \mathbb{F})) \equiv H_0(\mathbb{F} \times \mathbb{F})$ and $\varprojlim U_\alpha^i = U_\alpha$, the isomorphisms

$$\Psi_\alpha := \varprojlim \Psi_\alpha^i : p^{-1}(U_\alpha) \rightarrow U_\alpha \times H_0(\mathbb{F} \times \mathbb{F}), \quad a \in I,$$

can be defined, if $p = \varprojlim p^i$. These mappings provide a local topological trivialization on the limit $\varprojlim \mathcal{F}^2 M^i$ which can be thought of also as a differential one under the conventions of the Preliminaries Section referring to the generalized differential structure of $H_0(\mathbb{F} \times \mathbb{F})$. Then, we may easily check that each Ψ_α respects the action of $H_0(\mathbb{F} \times \mathbb{F})$ on the right of $\varprojlim \mathcal{F}^2 M^i$ defined by their counterparts on the factors in relation (2). As a result, $\varprojlim \mathcal{F}^2 M^i$ obtains a principal bundle structure over M with structure group $H_0(\mathbb{F} \times \mathbb{F})$ and transition functions satisfying

$$\Psi_\beta \circ \Psi_\alpha^{-1} = \varprojlim (\Psi_\beta^i \circ (\Psi_\alpha^i)^{-1})$$

□

We are now in a position to define the generalized bundle of frames of order two:

Definition 2.3. We call the *generalized bundle of frames of order two* of the Fréchet manifold $M = \varprojlim M^i$ the principal bundle

$$\mathcal{F}^2(M) := \varprojlim \mathcal{F}^2 M^i.$$

Remark 2.4. Summarizing some main properties of this new bundle we may note that:

- (i) The definition proposed is a natural generalization of the classical notion of second order frame bundle within the framework of Fréchet manifolds. Indeed, if M is a Banach modelled manifold, then the projective systems $\{M^i; \phi^{j^i}\}$, $\{\mathcal{F}^2 M^i; r^{j^i}\}$ reduce to the trivial ones $\{M; id_M\}$, $\{L^2 M; id_{L^2 M}\}$, where $L^2 M$ stands for the classical bundle of linear frames of $T^2 M$. Analogously, the topological group $H_0(\mathbb{F} \times \mathbb{F})$ coincides with the general linear group $GL(\mathbb{F} \times \mathbb{F})$. Thus, the limit $\varprojlim \mathcal{F}^2 M^i$ gives precisely the classical second order frame bundle of $M : \mathcal{F}^2 M \equiv L^2 M$.
- (ii) Based on the definition of the trivializations of $\mathcal{F}^2 M^i$, we may check that the transition functions of $\mathcal{F}^2 M$ take their values not on the entire structure group $H_0(\mathbb{F} \times \mathbb{F})$ but on a subgroup of it. Indeed, for every pair of indices $a, b \in I$, one obtains:

$$\begin{aligned} (\Psi_\beta \circ \Psi_\alpha^{-1})(x^i, (g^i)) &= \varprojlim ((\Psi_\beta^i \circ (\Psi_\alpha^i)^{-1})(x^i, (g^1, g^2, \dots, g^i))) \\ &= ((x^i), (\varprojlim T_{a\beta}^i((x^i)) \circ \varprojlim g^i)) = ((x^i), T_{a\beta}((x^i)) \circ \varprojlim g^i), \end{aligned}$$

where $\{T_{a\beta}\}_{a, \beta \in I}$ are the transition functions of the second order tangent bundle $T^2 M$ of M . However, as we may readily verify in view of (1), the second component $T_{a\beta}$ splits into two families of linear isomorphisms:

$$T_{a\beta} = (d(\psi_\beta \circ \psi_\alpha^{-1}) \circ \psi_\alpha) \times (d(\psi_\beta \circ \psi_\alpha^{-1}) \circ \psi_\alpha).$$

As a result, the transition functions of $\mathcal{F}^2 M$ take their values in

$$H_0(\mathbb{F}) \times H_0(\mathbb{F}) \subset H_0(\mathbb{F} \times \mathbb{F}).$$

- (iii) The generalized second order frame bundle \mathcal{F}^2M of M apart from being a projective limit of Banachable principal bundles, can be represented also in a form analogous to that of its factors:

$$\mathcal{F}^2M \equiv \bigcup_{x \in M} \{(h^i)_{i \in \mathbb{N}} \mid h^i \in \mathcal{L}is(\mathbb{F}^i \times \mathbb{F}^i, T_{\varphi^i(x)}^2 M^i) : \varprojlim h^i \text{ exists}\},$$

in view of the identification $((h^1, h^2, \dots, h^i))_{i \in \mathbb{N}} \equiv (h^i)_{i \in \mathbb{N}}$.

We have shown so far that for a wide class of Fréchet manifolds a generalized notion of the second order frame bundle can be defined and endowed with a principal bundle structure. The term *frame bundle* is further justified since the second order tangent bundle T^2M is associated with this new structure. In the next theorem we prove precisely this basic result.

Theorem 2.5. *For the action of the group $H^0(\mathbb{F} \times \mathbb{F})$ on the right of the product $\mathcal{F}^2(M) \times (\mathbb{F} \times \mathbb{F})$:*

$$((h^i), (u^i, v^i))_{i \in \mathbb{N}} \cdot (g^i)_{i \in \mathbb{N}} = ((h^i \circ g^i), (g^i)^{-1}(u^i, v^i))_{i \in \mathbb{N}},$$

the quotient space $\mathcal{F}^2M \times (\mathbb{F} \times \mathbb{F}) / H^0(\mathbb{F} \times \mathbb{F})$ is isomorphic with T^2M .

Proof. We denote by \tilde{E} the quotient under consideration and by

$$\tilde{\pi} : \tilde{E} \longrightarrow M : [(h^i), (u^i, v^i)] \mapsto p((h^i)) = \varprojlim(p^i(h^i))$$

its natural projection on M , where p and p^i are the projections of the bundles \mathcal{F}^2M , \mathcal{F}^2M^i to the corresponding bases. Again in an open smooth covering $\{(U_\alpha = \varprojlim U_\alpha^i, \psi_\alpha = \varprojlim \psi_\alpha^i)\}_{\alpha \in I}$ of M , and $\{\Psi_\alpha = \varprojlim \Psi_\alpha^i : p^{-1}(U_\alpha) \rightarrow U_\alpha \times H_0(\mathbb{F} \times \mathbb{F})\}_{\alpha \in I}$ the corresponding trivialisations of \mathcal{F}^2M obtained in Theorem 2.2, we may define, for each $\alpha \in I$, the mappings:

$$\tilde{\Phi}_\alpha : \tilde{\pi}^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{F} \times \mathbb{F} : [(h^i), (u^i, v^i)] \mapsto (p((h^i)), \Psi_{2,\alpha}((h^i))((u^i, v^i)))$$

where $\Psi_{2,\alpha}$ stands for the projection of Ψ_α to $H_0(\mathbb{F} \times \mathbb{F})$.

Each of these mappings is injective. Indeed, assuming that $\tilde{\Phi}_\alpha([(h^i), (u^i, v^i)]) = \tilde{\Phi}_\alpha([(h_1^i), (u_1^i, v_1^i)])$, we obtain first that $p((h^i)) = p((h_1^i)) = (x^i) \in M = \varprojlim M^i$. As a result, taking into account that $(h^i), (h_1^i) \in \mathcal{F}^2(M)$, we conclude that the limits $\varprojlim h^i$, $\varprojlim h_1^i$ can be defined. On the other hand, the equality between the second components of $\tilde{\Phi}_\alpha$ implies that

$$((\tau_\alpha^i \circ h^i)(u^i, v^i))_{i \in \mathbb{N}} = ((\tau_\alpha^i \circ h_1^i)(u_1^i, v_1^i))_{i \in \mathbb{N}},$$

where the linear isomorphisms $\tau_\alpha^i := pr_2 \circ \Phi_\alpha^i|_{(\pi_2^i)^{-1}(x^i)}$ have already been defined in Proposition 2.1. Thus, $(h^i(u^i, v^i))_{i \in \mathbb{N}} = (h_1^i(u_1^i, v_1^i))_{i \in \mathbb{N}}$. Considering the isomorphisms $g^i := (h_1^i)^{-1} \circ h^i \in GL(\mathbb{F}^i \times \mathbb{F}^i)$, the corresponding projective limit $\varprojlim g^i$ can be defined and, with respect to the element $g := (g^i)$ of $H^0(\mathbb{F} \times \mathbb{F})$, the classes $[(h^i), (u^i, v^i)]$ and $[(h_1^i), (u_1^i, v_1^i)]$ coincide.

Moreover, we can prove that $\tilde{\Phi}_\alpha$ is also onto. Namely, if $((x^i), (u^i, v^i))_{i \in \mathbb{N}}$ is an arbitrarily chosen point of $U_\alpha \times \mathbb{F} \times \mathbb{F}$, then setting $h^i := (\tau_\alpha^i)^{-1} \in \mathcal{L}is(\mathbb{F}^i \times \mathbb{F}^i, T_{x^i}^2 M^i)$, we easily check that $(h^i)_{i \in \mathbb{N}} \in \mathcal{F}^2(M)$ and

$$\tilde{\Phi}_\alpha([(h^i), (u^i, v^i)]) = ((x^i), ((\tau_\alpha^i \circ h^i)(u^i, v^i))) = ((x^i), (u^i, v^i)).$$

Considering now an arbitrary pair of indices $(\alpha, \beta) \in I^2$ and setting $\tilde{\Phi}_{\alpha,x} := pr_2 \circ \tilde{\Phi}_\alpha|_{\tilde{\pi}^{-1}(x)}$, we routinely check that the mappings

$$\tilde{\Phi}_{\beta,x} \circ \tilde{\Phi}_{\alpha,x}^{-1} : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F} \times \mathbb{F}$$

are linear isomorphisms since $\tilde{\Phi}_{\beta,x} \circ \tilde{\Phi}_{\alpha,x}^{-1} = \tau_{\beta,x} \circ \tau_{\alpha,x}^{-1}$.

A direct conclusion of the above line of thought is that \tilde{E} is a vector bundle over M with local trivializations the pairs $\{(U_\alpha, \tilde{\Phi}_\alpha)\}_{\alpha \in I}$ and transition functions

$$T_{\alpha\beta}(x) = \tilde{\Phi}_{\beta,x} \circ \tilde{\Phi}_{\alpha,x}^{-1} = \tau_{\beta,x} \circ \tau_{\alpha,x}^{-1}; \quad x \in U_\alpha \cap U_\beta.$$

In order now to prove that this bundle coincides, up to isomorphism, with the second order tangent bundle of M , we define the mapping

$$G : \tilde{E} \rightarrow T^2M : [(h^i), (u^i, v^i)] \mapsto (h^i(u^i, v^i)).$$

The latter can be defined since for every element $[(h^i), (u^i, v^i)]$ of the quotient \tilde{E} the mappings $\{h^i : \mathbb{F}^i \times \mathbb{F}^i \rightarrow T_{x^i}^2 M^i\}$ define a projective limit, $(u^i), (v^i)$ belong to $\mathbb{F} = \varprojlim \mathbb{F}^i$, hence the family $(h^i(u^i, v^i))$ is an element of $T^2M = \varprojlim T^2M^i$. Moreover, it is a well defined mapping since if the equivalence classes $[(h^i), (u^i, v^i)], [(h_1^i), (u_1^i, v_1^i)]$ coincide with respect to the element (g^i) of $H^0(\mathbb{F} \times \mathbb{F})$, then $h^i \circ g^i = h_1^i, g^i(u_1^i, v_1^i) = (u^i, v^i), i \in \mathbb{N}$, and

$$(h^i(u^i, v^i)) = (h^i(g^i(u_1^i, v_1^i))) = (h_1^i(u_1^i, v_1^i)).$$

On the other hand, G is one to one. Indeed, if $G([(h^i), (u^i, v^i)]) = G([(h_1^i), (u_1^i, v_1^i)])$, then $h^i(u^i, v^i) = h_1^i(u_1^i, v_1^i)$, for all $i \in I$, and the isomorphisms $g^i := (h_1^i)^{-1} \circ h^i$ can be defined. The corresponding projective limit exists, since the same holds true for both families (h^i) and (h_1^i) , and then

$$((h_1^i), (u_1^i, v_1^i)) \cdot (g^i) = ((h_1^i \circ g^i), (g^i)^{-1}(u_1^i, v_1^i)) = ((h^i), (u^i, v^i)),$$

thus obtaining $[(h^i), (u^i, v^i)] = [(h_1^i), (u_1^i, v_1^i)]$.

Moreover, G is also surjective: Let $(w^i) \in T^2M = \varprojlim T^2M^i$ with $w^i \in T_{x^i}^2 M^i, x = (x^i) \in M = \varprojlim M^i$. Considering a projective limit chart $(U_\alpha = \varprojlim U_\alpha^i, \psi_\alpha = \varprojlim \psi_\alpha^i)$ of M as well as its counterpart $(U_\alpha = \varprojlim U_\alpha^i, \Phi_\alpha = \varprojlim \Phi_\alpha^i)$ on T^2M (see also Preliminaries), we obtain the linear isomorphism $\tau_\alpha := pr_2 \circ \Phi_\alpha|_{\pi_2^{-1}(x)} = \varprojlim \tau_\alpha^i : T_x^2M \xrightarrow{\cong} \mathbb{F} \times \mathbb{F}$. Then, $((\tau_\alpha^i)^{-1})_{i \in \mathbb{N}} \in \mathcal{F}^2M, (\tau_\alpha^i(w^i))_{i \in \mathbb{N}} = (u^i, v^i)_{i \in \mathbb{N}}$ belongs to $\mathbb{F} \times \mathbb{F}$ and

$$G([((\tau_\alpha^i)^{-1})_{i \in \mathbb{N}}, (u^i, v^i)_{i \in \mathbb{N}}]) = ((\tau_\alpha^i)^{-1}(u^i, v^i))_{i \in \mathbb{N}} = (w^i)_{i \in \mathbb{N}}.$$

Finally, we observe that G is a vector bundle isomorphism since it maps the trivializations of \tilde{E} to those of T^2M :

$$\begin{aligned} (\Phi_\alpha \circ G)([(h^i)_{i \in \mathbb{N}}, (u^i, v^i)_{i \in \mathbb{N}}]) &= \Phi_\alpha((h^i(u^i, v^i))_{i \in \mathbb{N}}) = (\Phi_\alpha^i(h^i(u^i, v^i)))_{i \in \mathbb{N}} \\ &= (p((h^i)_{i \in \mathbb{N}}), ((\tau_\alpha^i \circ h^i)(u^i, v^i))_{i \in \mathbb{N}}) = \\ &= \tilde{\Phi}_\alpha([(h^i)_{i \in \mathbb{N}}, (u^i, v^i)_{i \in \mathbb{N}}]), \quad a \in I, \end{aligned}$$

thus concluding the proof. \square

3. GEOMETRIC ASSOCIATION OF THE SECOND ORDER BUNDLES

In view of the last theorem of the previous section, we may also proceed to an association of the connections of the generalized bundle of frames $\mathcal{F}^2(M)$ with the linear connections of the vector bundle T^2M .

We recall the classical relationship between connections of associated bundles. Namely, if we consider a connection of $\mathcal{F}^2(M)$ represented by the 1-form $\omega \in \Lambda^1(\mathcal{F}^2(M), \mathcal{L}(\mathbb{F} \times \mathbb{F}))$, a smooth atlas $\{(U_\alpha = \varprojlim U_\alpha^i, \psi_\alpha = \varprojlim \psi_\alpha^i)\}_{\alpha \in I}$ of M , $\{(p^{-1}(U_\alpha), \Psi_\alpha)\}_{\alpha \in I}$ the arising trivializations of $\mathcal{F}^2(M)$ and $\{\omega_\alpha := s_\alpha^* \omega\}_{\alpha \in I}$ the corresponding local forms of ω obtained as pull-backs with respect to the natural local sections $\{s_\alpha\}$ of $\{\Psi_\alpha\}$, then a (unique) linear connection can be defined on T^2M by means of the Christoffel symbols

$$\Gamma_\alpha : \psi_\alpha(U_\alpha) \rightarrow \mathcal{L}(\mathbb{F} \times \mathbb{F}, \mathcal{L}(\mathbb{F}, \mathbb{F} \times \mathbb{F}))$$

with $([\Gamma_\alpha(y)](u))(v) = \omega_\alpha(\psi_\alpha^{-1}(y))(T_y\psi_\alpha^{-1}(v))(u)$, $(y, u, v) \in \psi_\alpha(U_\alpha) \times \mathbb{F} \times \mathbb{F} \times \mathbb{F}$.

However, in the framework of Fréchet bundles an arbitrary connection is not always easy to handle, since Fréchet manifolds and bundles lack basic geometric properties underlying the cases of finite dimensional or Banach modelled bundles. In particular:

- Existence of parallel displacement along smooth curves of the base Fréchet manifold is problematic due to the lack of a general theory of solvability of linear differential equations.
- Handling corresponding Christoffel symbols (in the case of vector bundles) or the local forms (in principal bundles) is seriously affected by the fact that the space of continuous linear mappings of a Fréchet space does not remain in the same category.

Such difficulties can be solved if we focus our study on those connections that can be obtained as projective limits. A detailed study of such types of connections is presented in [11], [12]. In the case of the generalized second order frame bundle the following characterization holds:

Theorem 3.1. *Let ∇ be a linear connection of the second order tangent bundle $T^2M = \varprojlim T^2M^i$ that can be represented as a projective limit of linear connections ∇^i on the (Banach modelled) factors. Then ∇ corresponds to a connection form ω of \mathcal{F}^2M obtained also as a projective limit.*

Proof. Taking into account that the trivializations of both bundles T^2M and \mathcal{F}^2M are obtained as projective limits of the corresponding trivializations of their factors, we may proceed as follows:

Every connection $\nabla = \varprojlim \nabla^i$ is characterized by a family of Christoffel symbols $\{\Gamma_\alpha : \psi_\alpha(U_\alpha) \rightarrow \mathcal{L}(\mathbb{F} \times \mathbb{F}, \mathcal{L}(\mathbb{F}, \mathbb{F} \times \mathbb{F}))\}_{\alpha \in I}$ that are factorized into the form:

$$\Gamma_\alpha((y^i)_{i \in \mathbb{N}})((u^i, v^i)_{i \in \mathbb{N}}) = \varprojlim (\Gamma_\alpha^i(y^i)(u^i, v^i)),$$

where $\{\Gamma_\alpha^i : \psi_\alpha^i(U_\alpha^i) \rightarrow \mathcal{L}(\mathbb{F}^i \times \mathbb{F}^i, \mathcal{L}(\mathbb{F}^i, \mathbb{F}^i \times \mathbb{F}^i))\}_{\alpha \in I}$ are the Christoffel symbols of ∇^i . Connections ∇^i ($i \in \mathbb{N}$) correspond bijectively to a system of connection forms $\{\omega^i \in \Lambda^1(\mathcal{F}^2M^i, H_0^i(\mathbb{F} \times \mathbb{F}))\}_{i \in \mathbb{N}}$ whose projective limit $\omega := \varprojlim \omega^i \in \Lambda^1(\mathcal{F}^2M, H_0(\mathbb{F} \times \mathbb{F}))$ is the desired connection of the generalized second order frame bundle of M with corresponding local forms $\omega_\alpha((h^i)_{i \in \mathbb{N}}) = \varprojlim (\omega_\alpha^i(h^1, \dots, h^i))$, $\alpha \in I$. \square

4. AREAS FOR APPLICATION

Our constructions above have provided in the Fréchet manifold case a suitable bundle of frames \mathcal{F}^2M for the second tangent bundle T^2M , which is a vector bundle in the presence of a linear connection. Then T^2M is associated with \mathcal{F}^2M and a one to one correspondence between their connections is provided.

In a number of contexts, Fréchet spaces of sections arise as configurations of a physical field and then evolution equations necessarily involve second order differential operators. General references to geometric field theory include Albeverio et al. [2] and Deligne et al. [5]. Paycha [22] provides a summary of useful material and a substantial bibliography on geometric and operator methods in modern field theory, outlining approaches in Yang-Mills, Seiberg-Witten and string theory.

Here we mention several contexts where our new results may have a contribution to make by providing a suitable principal bundle for handling second tangent geometry.

- (1) The moduli space of inequivalent configurations is the quotient of the infinite-dimensional configuration space \mathcal{X} by the appropriate symmetry

gauge group. Typically, \mathcal{X} is modelled on a Fréchet space of smooth sections of a vector bundle over a closed manifold and is a Hilbert Lie group. Inverse limit Hilbert manifolds and inverse limit Hilbert groups, introduced by Omori [20, 21], provide an appropriate setting for the study of the Yang-Mills and Seiberg-Witten field equations.

- (2) Another area of application is the geometric setting of string theory (see Albeverio et al. [2], Deligne et al. [5], Nag and Sullivan [19]). Here Teichmüller theory is concerned with a configuration space of Riemannian metrics on a closed Riemann surface M . Then, \mathcal{X} is a space of smooth sections of $T^*M \otimes T^*M$ and it becomes an inverse limit Hilbert manifold. The universal Teichmüller space T_∞ , i.e. the inductive limit of the family of Teichmüller spaces on each surface, can be obtained as the projective limit of all finite sheeted compact unbranched coverings of a given closed Riemann surface M of genus $g \geq 2$. This is a universal object, called the universal hyperbolic solenoid which can parametrize complex structures on surfaces of all topologies.
- (3) Let M be a finite-dimensional path-connected Riemannian manifold. The space of all smooth maps from the circle group S^1 to M is the Fréchet manifold ΛM called the space of free loops in M . Manoharan [17, 18] has provided a number of results on ΛM . A string structure is defined as a lifting of the structure group to an S^1 -central extension of the loop group. Suppose that $\tilde{G} \rightarrow \tilde{P} \rightarrow X$ is a lifting of a principal Fréchet bundle $G \rightarrow P \rightarrow X$ over a Fréchet manifold X and further that $S^1 \rightarrow \tilde{G} \rightarrow G$ is an S^1 -central extension of G . Manoharan showed that every connection on the principal bundle $G \rightarrow P \rightarrow X$ together with a \tilde{G} -invariant connection on $S^1 \rightarrow \tilde{P} \rightarrow P$ defines a connection on $\tilde{G} \rightarrow \tilde{P} \rightarrow X$. Hence there exist connections on the string structure of ΛM .
- (4) The group \mathcal{D} of orientation preserving smooth diffeomorphisms of a compact manifold M is homeomorphic to the product of the group of volume preserving diffeomorphisms \mathcal{D}_μ , of a volume element μ on M , times the set \mathcal{V} of all volumes $v > 0$ with $\int v = \int \mu$. In this case, \mathcal{D}_μ can be realized as a projective limit of Hilbert-modelled manifolds (see Omori [20, 21]) and forms the appropriate framework for the study of hydrodynamics of an incompressible fluid. More precisely, the motion of a perfect incompressible fluid is a geodesic curve η_t of \mathcal{D}_μ with respect to the right invariant metric on \mathcal{D}_μ , which at $e \in \mathcal{D}_\mu$ is $(X, Y) = \int_M \langle X(m), Y(m) \rangle_m \mu(m)$.

Moreover, there is a close relationship between geodesics on \mathcal{D}_μ and the classical Euler equations for a perfect fluid. Namely, if $\eta_t \in \mathcal{D}_\mu$ is a geodesic of \mathcal{D}_μ as above and $v_t = d\eta_t/dt$ the velocity, then the vector field $u_t = v_t \circ \eta_t^{-1}$ of M is a solution to the classical Euler equations

$$\begin{aligned} \frac{\partial u_t}{\partial t} + \nabla_{u_t} u_t &= \text{grad} p_t, \\ \text{div} u_t &= 0, \\ u_t \text{ given at } t &= 0, \\ u_t \text{ tangent to } \partial M, & \end{aligned}$$

Here p_t stands for the pressure and ∇ for the covariant derivative. Details can be found in [9].

For recent results see also Golovin [13], who calculated bases of differential invariants for infinite dimensional Lie groups, admitted by the Navier-Stokes and gas dynamics equations. He provided examples of the

group stratification for the stationary gas dynamics and for the transonic gas motion equations.

- (5) The space $J^\infty E$ of infinite jets of the sections of a Banach modelled vector bundle E can be realized as the projective limit of the finite corresponding jets $\{J^k E\}_{k \in \mathbb{N}}$. This approach makes possible the definition of a Fréchet modelled vector bundle on $J^\infty E$ and thus the use of the latter for the description of Lagrangians and source equations as certain types of differential forms (see Galanis [11] and Takens [24]).

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