Some recent work in Fréchet geometry*

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Abstract

Fréchet spaces of sections arise naturally as configurations of a physical field. For this presentation some recent work in Fréchet geometry is briefly reviewed, cf. Dodson [5] for some more details.

An earlier result on the structure of second tangent bundles in the finite dimensional case was extended to infinite dimensional Banach manifolds and Fréchet manifolds that could be represented as projective limits of Banach manifolds. This led to further results concerning the characterization of second tangent bundles and differential equations in the more general Fréchet structure needed for applications.

Introduction

Dodson and Radivoiovici [11, 12] proved that in the case of a finite n-dimensional manifold M, a vector bundle structure on T^2M can be well defined if and only if M is endowed with a linear connection: T^2M becomes then and only then a vector bundle over M with structure group the general linear group $GL(2n;\mathbb{R})$.

The manifolds M that admit linear connections are precisely the paracompact ones. Manifolds with linear connections form a full subcategory $Man\nabla$ of the category Man of smooth manifolds and smooth maps; the constructions in the above theorems [11] provide a functor $Man\nabla \longrightarrow VBun$ [12].

A linear connection is a splitting of TLM, which then induces splitting in the second jet bundle J^2M (called a *dissection* by Ambrose et al. [3]) and we get also a corresponding splitting in T^2L^2M.

Dodson and Galanis [6] extended the results to manifolds M modeled on an arbitrarily chosen Banach space $\mathbb{E}$. Using the Vilms [29] point of view for connections on infinite dimensional vector bundles and a new formalism, it was proved that T^2M can be thought of as a Banach vector bundle over M with structure group $GL(\mathbb{E} \times \mathbb{E})$ if and only if M admits a linear connection.

The case of non-Banach Fréchet modeled manifolds was investigated [6] but there are intrinsic difficulties with Fréchet spaces. These include pathological general linear groups, which do not even admit reasonable topological group structures.

However, every Fréchet space admits representation as a projective limit of Banach spaces and under certain conditions this can persist into manifold structures.

By restriction to Fréchet manifolds obtained as projective limits of Banach manifolds [13], T^2M gains a vector bundle structure over M with structure group a new topological group, that in a generalized sense is of Lie type.

This construction is equivalent to the existence on M of a specific type of linear connection characterized by a generalized set of Christoffel symbols.

What makes the Fréchet case important but difficult?

In significant cases in global analysis and physical field theory, Banach space representations break down and we need Fréchet spaces, with weaker requirements for their topology.

See for example Clarke [4] for the metric geometry of the Fréchet manifold of all C^∞ Riemannian metrics on a fixed closed finite-dimensional orientable manifold.

For background to Fréchet space theory see Hamilton [19] and Neeb [22], Steen and Seebach [26].

However, there is a price to pay for these weaker structural constraints: Fréchet spaces lack a general solvability theory of differential equations, even linear ones; also, the space of continuous linear mappings drops out of the Fréchet category while the space of linear isomorphisms does not admit a reasonable Lie group structure.

We shall see that these problems can be worked round to a certain extent.

The developments described in this short review will be elaborated in detail in the forthcoming monograph by Dodson, Galanis and Vassilliou [10].

Fréchet spaces

A *seminorm* on (eg for definiteness a real) vector space X is a real valued map $p : X \to \mathbb{R}$ such that

$$p(x) \geq 0, \tag{i}$$
$$p(x+y) \leq p(x) + p(y), \tag{ii}$$
$$p(\lambda x) = |\lambda|\, p(x), \ \ \forall x, y \in X, \text{ and } \lambda \in \mathbb{R}. \tag{iii}$$

A family of seminorms $\Gamma = \{p_\alpha\}_{\alpha \in I}$ on X defines a unique topology $\mathcal{T}_\Gamma$ compatible with the vector structure of X.

Neighborhood base $\mathcal{B}_\Gamma$ of $\mathcal{T}_\Gamma$ is determined by the family

$$S(\Delta, \varepsilon) = \{x \in \mathbb{F} : p(x) < \varepsilon, \ \forall\, p \in \Delta\}.$$

$$\mathcal{B}_\Gamma = \{S(\Delta, \varepsilon) : \varepsilon > 0 \text{ and } \Delta \text{ a finite subset of } \Gamma\},$$

Induced topology $\mathcal{T}_\Gamma$ on X by p is largest making all seminorms continuous but is not necessarily Hausdorff. $(X, \mathcal{T}_\Gamma)$ is a locally convex topological vector space and local convexity of topology on X is its subordination to a family of seminorms.

Hausdorffness: requires the further property

$$x = 0 \Leftrightarrow p(x) = 0, \ \forall p \in \Gamma.$$

Then it is metrizable if and only if Γ is countable.

Convergence: of $(x_n)_{n\in\mathbb{N}}$ in X depends on all of Γ

$$x_n \to x \Leftrightarrow \ p(x_n - x) \to 0, \ \forall p \in \Gamma.$$

Completeness: if and only if convergence in X of every

$$(x_n)_{n\in\mathbb{N}} \in X \text{ with } \lim_{n.m\to\infty} p(x_n - x_m) = 0; \ \forall \, p \in \Gamma.$$

Definition

A *Fréchet space* is a topological vector space $\mathbb{F}$ that is locally convex, Hausdorff, metrizable and complete.

So, every Banach space is a Fréchet space, with just one seminorm and that one is a norm. More interesting examples include the following:

- The space $\mathbb{R}^\infty = \prod_{n\in\mathbb{N}} \mathbb{R}^n$, endowed with the cartesian topology, is a Fréchet space with corresponding family of seminorms

$$\{p_n(x_1, x_2, ...) = |x_1| + |x_2| + ... + |x_n|\}_{n\in\mathbb{N}}.$$

  Metrizability can be established by putting

$$d(x, y) = \sum_i \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)}.$$

In $\mathbb{R}^\infty$ the completeness is inherited from that of each copy of the real line. For if $x = (x_i)$ is Cauchy in $\mathbb{R}^\infty$ then for each i, $(x_i^m), m \in \mathbb{N}$ is Cauchy in $\mathbb{R}$ and hence converges, to X_i say, and $(X_i) = X \in \mathbb{R}^\infty$ with $d(x_i, X_i) \to 0$ as $i \to \infty$.

Separability arises from the countable dense subset of elements having finitely many rational components and the remainder zero; second countability comes from metrizability.

Hausdorfness implies that a compact subset of a Fréchet space is closed; a closed subspace is a Fréchet space and a quotient by a closed subspace is a Fréchet space.

$\mathbb{R}^\infty$ is a special case from a classification for Fréchet spaces [22]. For each seminorm $p_n = || \ ||_n$ define the normed subspace $F_n = F/p_n^{-1}(0)$ by factoring out the null space of p_n.

Then the seminorm requirement provides a linear injection into the product of normed spaces

$$p : F \to \prod_{n \in \mathbb{N}} F_n : f \mapsto (p_n(f))_{n \in \mathbb{N}}$$

and completeness of F is equivalent to closedness of $p(F)$ in the Banach product of the closures $\overline{F_n}$ and p extends to an embedding of F in this product.

This embedding yields limiting processes for geometric structures in Fréchet manifolds modelled on F.

- More generally, any *countable* cartesian product of Banach spaces $\mathbb{F} = \prod_{n \in \mathbb{N}} \mathbb{E}^n$ is a Fréchet space with topology defined by the seminorms $(q_n)_{n \in \mathbb{N}}$, given by

$$q_n(x_1, x_2, \ldots) = \sum_{i=1}^{n} \|x_i\|_i \,, \quad \text{where } \|\cdot\|_i \text{ is norm on } \mathbb{E}^i.$$

- The space of continuous functions $C^0(\mathbb{R}, \mathbb{R})$ is a Fréchet space with seminorms $(p_n)_{n \in \mathbb{N}}$ defined by

$$p_n(f) = \sup \left\{ |f(x)| \,,\; x \in [-n, n] \right\}.$$

- The space of smooth functions $C^\infty(I, \mathbb{R})$, where I is a compact interval of $\mathbb{R}$, is a Fréchet space with seminorms defined by

$$p_n(f) = \sum_{i=0}^{n} \sup \left\{ \left| D^i f(x) \right| ,\; x \in I \right\}.$$

- The space $C^\infty(M, V)$, of smooth sections of the vector bundle V over compact smooth Riemannian manifold M with covariant derivative ∇, is a Fréchet space with

$$||f||_n = \sum_{i=0}^{n} sup_x |\nabla^i f(x)|, \quad \text{for } n \in \mathbb{N}.$$

- Fréchet spaces of sections arise naturally as configurations of a physical field. Then the moduli space, consisting of inequivalent configurations of the physical field, is the quotient of the infinite-dimensional configuration space $\mathcal{X}$ by the appropriate symmetry gauge group.

  Typically, the configuration space $\mathcal{X}$ is modelled on a Fréchet space of smooth sections of a vector bundle over a closed manifold. For example, see Omori [23, 24] and Clarke [4].

Banach second tangent bundle

Let M be a $C^\infty-$manifold modeled on Banach space $\mathbb{E}$ with atlas $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in I}$.
It induces atlas $\{(\pi_M^{-1}(U_\alpha), \Psi_\alpha)\}_{\alpha\in I}$ of TM with

$$\Psi_\alpha : \pi_M^{-1}(U_\alpha) \longrightarrow \psi_\alpha(U_\alpha) \times \mathbb{E} : [c, x] \longmapsto (\psi_\alpha(x), (\psi_\alpha \circ c)'(0)),$$

where $[c, x]$ stands for the equivalence class of a smooth curve c of M with $c(0) = x$ and

$$(\psi_\alpha \circ c)'(0) = [d(\psi_\alpha \circ c)(0)](1).$$

The corresponding trivializing system of $T(TM)$ is denoted by

$$\{(\pi_{TM}^{-1}(\pi_M^{-1}(U_\alpha)), \widetilde{\Psi}_\alpha)\}_{\alpha\in I}.$$

Adopting the formalism of Vilms [29], a connection on M is a vector bundle morphism: $\nabla : T(TM) \longrightarrow TM$.

∇ has the additional property that the mappings $\omega_\alpha : \psi_\alpha(U_\alpha) \times \mathbb{E} \to \mathcal{L}(\mathbb{E}, \mathbb{E})$ defined by the local forms of ∇ :

$$\nabla_\alpha : \psi_\alpha(U_\alpha) \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \to \psi_\alpha(U_\alpha) \times \mathbb{E}$$

with $\nabla_\alpha := \Psi_\alpha \circ \nabla \circ (\widetilde{\Psi}_\alpha)^{-1}$, $\alpha \in I$, via the relation

$$\nabla_\alpha(y, u, v, w) = (y, w + \omega_\alpha(y, u) \cdot v),$$

are smooth. Furthermore, ∇ is a linear connection on M if and only if $\{\omega_\alpha\}_{\alpha \in I}$ are linear with respect to the second variable.

Such a connection ∇ is fully characterized by the family of Christoffel symbols $\{\Gamma_\alpha\}_{\alpha \in I}$, which are smooth mappings

$$\Gamma_\alpha : \psi_\alpha(U_\alpha) \longrightarrow \mathcal{L}(\mathbb{E}, \mathcal{L}(\mathbb{E}, \mathbb{E}))$$

defined by $\Gamma_\alpha(y)[u] = \omega_\alpha(y, u)$, $(y, u) \in \psi_\alpha(U_\alpha) \times \mathbb{E}$.

These Christoffel symbols satisfy the compatibility condition:

$$\Gamma_\alpha(\sigma_{\alpha\beta}(y))(d\sigma_{\alpha\beta}(y)(u))[d(\sigma_{\alpha\beta}(y))(v)] + (d^2\sigma_{\alpha\beta}(y)(v))(u) = \\ = d\sigma_{\alpha\beta}(y)((\Gamma_\beta(y)(u))(v)),$$

for all $(y, u, v) \in \psi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{E} \times \mathbb{E}$, and d, d^2 are first and second differential. $\sigma_{\alpha\beta}$ denotes the diffeomorphisms $\psi_\alpha \circ \psi_\beta^{-1}$ of $\mathbb{E}$.

For further details and the relevant proofs see Vilms [29].

Take smooth M modeled on Banach $\mathbb{E}$ with atlas $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$.
For each $x \in M$ define equivalence relation on
$C_x = \{f : (-\varepsilon, \varepsilon) \to M \mid f$ smooth and $f(0) = x, \varepsilon > 0\}$:

$$f \approx_x g \Leftrightarrow f'(0) = g'(0) \text{ and } f''(0) = g''(0),$$

where by f' and f'' we denote, respectively, the first and the second derivatives of f :

$$\begin{aligned} f' &: (-\varepsilon, \varepsilon) \to TM : t \longmapsto [df(t)](1) \\ f'' &: (-\varepsilon, \varepsilon) \to T(TM) : t \longmapsto [df'(t)](1). \end{aligned}$$

The *tangent space of order two* of M at the point x is the quotient $T_x^2 M = C_x / \approx_x$.

The *tangent bundle of order two* of M is the union of all tangent spaces of order 2: $T^2 M := \bigcup_{x \in M} T_x^2 M$.

Of course, $T^2_x M$ can be thought of as a topological vector space isomorphic to $\mathbb{E} \times \mathbb{E}$ via the bijection

$$T^2_x M \xleftrightarrow{\simeq} \mathbb{E} \times \mathbb{E} : [f, x]_2 \longmapsto ((\psi_\alpha \circ f)'(0), (\psi_\alpha \circ f)''(0)),$$

where $[f, x]_2$ is the equivalence class of f with respect to $\approx_x$. However, that structure depends on the choice of the chart (U_α, ψ_α), hence a definition of a vector bundle structure on $T^2 M$ cannot be achieved by the use of the above bijections.

The geometric way to overcome this obstacle is to assume that the manifold M is endowed with the additional structure of a linear connection, Dodson and Galanis [6], following the finite-dimensional method of Dodson and Radivoiovici [11].

Theorem ([6])

For every linear connection ∇ on the Banach manifold M, $T^2 M$ becomes a Banach vector bundle with structure group the general linear group $GL(\mathbb{E} \times \mathbb{E})$.

Proof

Let $\pi_2 : T^2M \to M$ be the natural projection of T^2M to M with $\pi_2([f,x]_2) = x$ and $\{\Gamma_\alpha : \psi_\alpha(U_\alpha) \longrightarrow \mathcal{L}(\mathbb{E}, \mathcal{L}(\mathbb{E}, \mathbb{E}))\}_{a \in I}$ the Christoffel symbols of the connection D with respect to the covering $\{(U_a, \psi_a)\}_{a \in I}$ of M. Then, for each $\alpha \in I$, we define the mapping $\Phi_\alpha : \pi_2^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{E} \times \mathbb{E}$ with

$$\Phi_\alpha([f,x]_2) = (x, (\psi_\alpha \circ f)'(0), (\psi_\alpha \circ f)''(0)$$

$$+\Gamma_\alpha(\psi_\alpha(x))((\psi_\alpha \circ f)'(0))[(\psi_\alpha \circ f)'(0)]).$$

Well defined and injective, these are also surjective since every $(x, u, v) \in U_\alpha \times \mathbb{E} \times \mathbb{E}$ can be obtained through Φ_α as image of the equivalence class of the smooth curve

$$f : \mathbb{R} \to \mathbb{E} : t \mapsto \psi_\alpha(x) + tu + \frac{t^2}{2}(v - \Gamma_\alpha(\psi_\alpha(x))(u)[u]),$$

appropriately restricted in order to take values in $\psi_\alpha(U_\alpha)$. On the other hand, the projection of each Φ_α to the first factor coincides with the natural projection $\pi_2 : pr_1 \circ \Phi_\alpha = \pi_2$.

Therefore, the trivializations $\{(U_\alpha, \Phi_\alpha)\}_{a\in I}$ define a fibre bundle structure on T^2M and we need to consider change of chart maps.

If (U_α, ψ_α), (U_β, ψ_β) are two overlapping charts, let $(\pi_2^{-1}(U_\alpha), \Phi_\alpha)$, $(\pi_2^{-1}(U_\beta), \Phi_\beta)$ be the corresponding trivializations of T^2M. Taking into account the compatibility condition above satisfied by the Christoffel symbols $\{\Gamma_\alpha\}$ we see that

$$(\Phi_\alpha \circ \Phi_\beta^{-1})(x, u, v) = \Phi_\alpha([f, x]_2)$$

where $(\psi_\beta \circ f)'(0) = u$ and $(\psi_\beta \circ f)''(0) + \Gamma_\beta(\psi_\beta(x))(u)[u] = v$.

After some algebra,

$$(\Phi_\alpha \circ \Phi_\beta^{-1})(x, u, v) = (\sigma_{\alpha\beta}(\psi_\beta(x)), d\sigma_{\alpha\beta}(\psi_\beta(x))(u), d\sigma_{\alpha\beta}(\psi_\beta(x))(v)),$$

where $\sigma_{\alpha\beta}$ are the diffeomorphisms $\psi_\alpha \circ \psi_\beta^{-1}$.

Hence, the restrictions to the fibres

$$\Phi_{\alpha,x} \circ \Phi_{\beta,x}^{-1} : \mathbb{E} \times \mathbb{E} \to \mathbb{E} \times \mathbb{E} :(u, v) \longmapsto (\Phi_\alpha \circ \Phi_\beta^{-1})|_{\pi_2^{-1}(x)}(u, v)$$

are linear isomorphisms and the mappings:

$$T_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathcal{L}(\mathbb{E} \times \mathbb{E}, \mathbb{E} \times \mathbb{E}) : x \longmapsto \Phi_{\alpha,x} \circ \Phi_{\beta,x}^{-1}$$

are smooth since $\forall \alpha, \beta \in I$, $T_{\alpha\beta} = (d\sigma_{\alpha\beta} \circ \psi_\beta) \times (d\sigma_{\alpha\beta} \circ \psi_\beta)$.

As a result, T^2M is a vector bundle over M with fibres of type $\mathbb{E} \times \mathbb{E}$ and structure group $GL(\mathbb{E} \times \mathbb{E})$.
T^2M is isomorphic to $TM \times TM$ since both bundles are characterized by the same transition function cocycle $\{(d\sigma_{\alpha\beta} \circ \psi_\beta) \times (d\sigma_{\alpha\beta} \circ \psi_\beta)\}_{\alpha,\beta \in I}$. □

The converse of this theorem was proved also in [6].

These theorems coincide in the finite dimensional case with the earlier results since the corresponding transition functions are identical (see Dodson and Radivoiovici [11] Corollary 2).

The results [11, 12] on the *frame bundle of order two*

$$L^2(M) := \bigcup_{x\in M} \mathcal{L}is(\mathbb{E}\times\mathbb{E}, T_x^2 M),$$

were extended also to the Banach manifold M by Dodson and Galanis:

Theorem ([7])

Every linear connection ∇ on the second order tangent bundle T^2M on the Banach manifold M corresponds bijectively to a connection ω of $L^2(M)$.

Fréchet second tangent bundle

Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be two *Hausdorff locally convex topological vector spaces*, and let U be an open subset of $\mathbb{F}_1$. A continuous map $f : U \to \mathbb{F}_2$ is called *differentiable at* $x \in U$ if there exists a continuous linear map $Df(x) : \mathbb{F}_1 \to \mathbb{F}_2$ such that

$$R(t,v) := \begin{cases} \frac{1}{t}\left(f(x+tv) - f(x) - Df(x)(tv)\right), & t \neq 0 \\ 0, & t = 0 \end{cases}$$

is continuous at every $(0, v) \in \mathbb{R}\times\mathbb{F}_1$. The map f will be said to be *differentiable* if it is differentiable at every $x \in U$.

We call $Df(x)$ the *differential* (or *derivative*) *of f at x*.
As in classical (Fréchet) differentiation, $Df(x)$ is uniquely determined; see Leslie [20] and [21] for more details.
A map $f : U \to \mathbb{F}_2$, as before, is called C^1*-differentiable* if it is differentiable at every point $x \in U$, and we have continuity of the *(total) differential* or *(total) derivative*

$$Df : U \times \mathbb{F}_1 \to \mathbb{F}_2 : (x, v) \mapsto Df(x)(v)$$

This Df does not involve the space of continuous linear maps $\mathcal{L}(\mathbb{F}_1, \mathbb{F}_2)$, avoiding the possibility of dropping out of the Fréchet category when $\mathbb{F}_1$ and $\mathbb{F}_2$ are Fréchet spaces.
C^n-differentiability ($n \geq 2$) can be defined by induction and C^∞-differentiability follows.

Galanis and Vassiliou [14, 28] for tangent and frame bundles yields a vector bundle structure on second order tangent bundles for Fréchet manifolds obtainable as projective limits of Banach manifolds, Dodson and Galanis [6].

Let M be a smooth manifold modeled on the Fréchet space $\mathbb{F}$. Taking into account that the latter *always* can be realized as a projective limit of Banach spaces $\{\mathbb{E}^i; \rho^{ji}\}_{i,j\in\mathbb{N}}$ (i.e. $\mathbb{F} \cong \varprojlim \mathbb{E}^i$) we assume that the manifold itself is obtained as the limit of a projective system of Banach modeled manifolds $\{M^i; \varphi^{ji}\}_{i,j\in\mathbb{N}}$.

Then, it was proved [6] that the second order tangent bundles $\{T^2 M^i\}_{i\in\mathbb{N}}$ form also a projective system with limit (set-theoretically) isomorphic to $T^2 M$.

We define a vector bundle structure on $T^2 M$ by means of a certain type of linear connection on M.
Problems with the structure group of this bundle are overcome by replacing the pathological $GL(\mathbb{F} \times \mathbb{F})$ by the new topological ('generalized smooth' Lie) group:

$$\mathcal{H}^0(\mathbb{F} \times \mathbb{F}) := \{(l^i)_{i\in\mathbb{N}} \in \prod_{i=1}^{\infty} GL(\mathbb{E}^i \times \mathbb{E}^i) : \varprojlim l^i \text{ exists}\}.$$

Precisely, $\mathcal{H}^0(\mathbb{F}\times\mathbb{F})$ is a topological group that is isomorphic to the projective limit of the Banach-Lie groups

$$\mathcal{H}_i^0(\mathbb{F}\times\mathbb{F}) := \{(l^1, l^2, ..., l^i)_{i\in\mathbb{N}} \in \prod_{k=1}^{i} GL(\mathbb{E}^k\times\mathbb{E}^k)\} : \rho^{jk}\circ l^j = l^k\circ\rho^{jk} \ (k\leq j\leq i).$$

Also, it can be considered as a generalized Lie group via its embedding in the topological vector space $\mathcal{L}(\mathbb{F}\times\mathbb{F})$.

Theorem ([6])

If a Fréchet manifold $M=\varprojlim M^i$ is endowed with a linear connection ∇ that can be realized also as the projective limit of connections $\nabla=\varprojlim\nabla^i$, then T^2M is a Fréchet vector bundle over M with structure group $\mathcal{H}^0(\mathbb{F}\times\mathbb{F})$.

Proof

Following the terminology established above, we consider $\{(U_\alpha = \varprojlim U^i_\alpha, \psi_\alpha = \varprojlim \psi^i_\alpha)\}_{\alpha\in I}$ an atlas of M. Each linear connection ∇^i ($i \in \mathbb{N}$), which is naturally associated to a family of Christoffel symbols $\{\Gamma^i_\alpha : \psi^i_\alpha(U^i_\alpha) \to \mathcal{L}(\mathbb{E}^i, \mathcal{L}(\mathbb{E}^i, \mathbb{E}^i))\}_{\alpha\in I}$, ensures that T^2M^i is a vector bundle over M^i with fibres of type $\mathbb{E}^i$. We defined this structure above, with trivializations:

$$\Phi^i_\alpha : (\pi^i_2)^{-1}(U^i_\alpha) \longrightarrow U^i_\alpha \times \mathbb{E}^i \times \mathbb{E}^i.$$

The families of mappings $\{g^{ji}\}_{i,j\in\mathbb{N}}$, $\{\varphi^{ji}\}_{i,j\in\mathbb{N}}$, $\{\rho^{ji}\}_{i,j\in\mathbb{N}}$ are connecting morphisms of the projective systems
$T^2M = \varprojlim(T^2M^i)$, $M = \varprojlim M^i$, $\mathbb{F} = \varprojlim \mathbb{E}^i$.
These projections $\{\pi^i_2 : T^2M^i \to M^i\}_{i\in\mathbb{N}}$ and trivializations $\{\Phi^i_\alpha\}_{i\in\mathbb{N}}$ satisfy

$$\begin{aligned} \varphi^{ji} \circ \pi^j_2 &= \pi^i_2 \circ g^{ji} \quad (j \geq i) \\ (\varphi^{ji} \times \rho^{ji} \times \rho^{ji}) \circ \Phi^j_\alpha &= \Phi^i_\alpha \circ g^{ji} \quad (j \geq i). \end{aligned}$$

We obtain the surjection $\pi_2 = \varprojlim \pi_2^i : T^2M \longrightarrow M$ and,

$$\Phi_\alpha = \varprojlim \Phi_\alpha^i : \pi_2^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{F} \times \mathbb{F} \ \ (\alpha \in I)$$

is smooth, as a projective limit of smooth mappings, and its projection to the first factor coincides with π_2. The restriction to a fibre $\pi_2^{-1}(x)$ of Φ_α is a bijection since
$\Phi_{\alpha,x} := pr_2 \circ \Phi_\alpha|_{\pi_2^{-1}(x)} = \varprojlim (pr_2 \circ \Phi_\alpha^i|_{(\pi_2^i)^{-1}(x)})$.
The corresponding transition functions $\{T_{\alpha\beta} = \Phi_{\alpha,x} \circ \Phi_{\beta,x}^{-1}\}_{\alpha,\beta \in I}$ can be considered as taking values in the generalized Lie group $\mathcal{H}^0(\mathbb{F} \times \mathbb{F})$, since $T_{\alpha\beta} = \epsilon \circ T_{\alpha\beta}^*$, where $\{T_{\alpha\beta}^*\}_{\alpha,\beta \in I}$ are the smooth mappings

$$T_{\alpha\beta}^* : U_\alpha \cap U_\beta \to \mathcal{H}^0(\mathbb{F} \times \mathbb{F}) : x \longmapsto (pr_2 \circ \Phi_\alpha^i|_{(\pi_2^i)^{-1}(x)})_{i \in \mathbb{N}}$$

with ϵ the natural inclusion

$$\epsilon : \mathcal{H}^0(\mathbb{F} \times \mathbb{F}) \to \mathcal{L}(\mathbb{F} \times \mathbb{F}) : (l^i)_{i \in \mathbb{N}} \longmapsto \varprojlim l^i.$$

Hence, T^2M admits a vector bundle structure over M with fibres of type $\mathbb{F} \times \mathbb{F}$ and structure group $\mathcal{H}^0(\mathbb{F} \times \mathbb{F})$.
This bundle is isomorphic to $TM \times TM$ since they have identical transition functions:

$$T_{\alpha\beta}(x) = \Phi_{\alpha,x} \circ \Phi_{\beta,x}^{-1} = (d(\psi_a \circ \psi_\beta^{-1}) \circ \psi_\beta)(x) \times (d(\psi_a \circ \psi_\beta^{-1}) \circ \psi_\beta)(x)$$

□

Also, the converse is true:

Theorem ([6])

If T^2M is an $\mathcal{H}^0(\mathbb{F} \times \mathbb{F})-$Fréchet vector bundle over M isomorphic to $TM \times TM$, then M admits a linear connection which can be realized as a projective limit of connections.

Fréchet second frame bundle

Let $M = \varprojlim M^i$ be a manifold with connecting morphisms $\{\varphi^{ji} : M^j \to M^i\}_{i,j\in\mathbb{N}}$ and Fréchet space model the limit $\mathbb{F}$ of a projective system of Banach spaces $\{\mathbb{F}^i; \rho^{ji}\}_{i,j\in\mathbb{N}}$. Following the results obtained in [6], if M is endowed with a linear connection $\nabla = \varprojlim \nabla^i$, then T^2M admits a vector bundle structure over M with fibres of Fréchet type $\mathbb{F}\times\mathbb{F}$. Then T^2M becomes also a projective limit of manifolds via the identification $T^2M \simeq \varprojlim T^2M^i$.

Using the connecting morphisms $\{g^{ji}\}_{i,j\in\mathbb{N}}$ of the projective systems $T^2M = \varprojlim(T^2M^i)$, consider the sequences of linear isomorphisms

$$\mathcal{F}^2M^i = \bigcup_{x^i\in M^i}\{(h^k)_{k=1,\dots,i} : h^k \in \mathcal{L}is(\mathbb{F}^k\times\mathbb{F}^k, T^2_{\varphi^{ik}(x^i)}M^k)$$

$$\text{with} \quad g^{mk}\circ h^m = h^k\circ(\rho^{mk}\times\rho^{mk}),\ i\geq m\geq k\}.$$

We replace the pathological general linear group $GL(\mathbb{F})$ by

$$H_0(\mathbb{F}) := H_0(\mathbb{F},\mathbb{F}) = \{(l^i)_{i\in\mathbb{N}} \in \prod_{i=1}^{\infty} GL(\mathbb{F}^i) : \varprojlim l^i \text{ exists}\}.$$

The latter can be thought of also as a generalized Fréchet Lie group by being embedded in $H(\mathbb{F}) := H(\mathbb{F},\mathbb{F})$, from Dodson, Galanis and Vassiliou [8]:

Theorem ([8])

$\mathcal{F}^2 M^i$ is a principal fibre bundle over M^i with structure group the Banach Lie group $H_0^i(\mathbb{F}\times\mathbb{F}) := H_0^i(\mathbb{F}\times\mathbb{F},\mathbb{F}\times\mathbb{F})$. The limit $\varprojlim \mathcal{F}^2 M^i$ is a Fréchet principal bundle over M with structure group $H_0(\mathbb{F}\times\mathbb{F})$.

We call the *generalized bundle of frames of order two* of the Fréchet manifold $M = \varprojlim M^i$ the principal bundle

$$\mathcal{F}^2(M) := \varprojlim \mathcal{F}^2 M^i.$$

This is a natural generalization of the usual frame bundle and from Dodson, Galanis and Vassiliou [8] we have:

Theorem ([8])

For the action of the group $H^0(\mathbb{F}\times\mathbb{F})$ on the right of the product $\mathcal{F}^2(M)\times(\mathbb{F}\times\mathbb{F})$:

$$((h^i),(u^i,v^i))_{i\in\mathbb{N}}\cdot(g^i)_{i\in\mathbb{N}}=((h^i\circ g^i),(g^i)^{-1}(u^i,v^i))_{i\in\mathbb{N}},$$

the quotient space $\mathcal{F}^2M\times(\mathbb{F}\times\mathbb{F})\big/H^0(\mathbb{F}\times\mathbb{F})$ is isomorphic with T^2M.

Consider a connection of $\mathcal{F}^2(M)$ represented by the 1-form $\omega\in\Lambda^1(\mathcal{F}^2(M),\mathcal{L}(\mathbb{F}\times\mathbb{F}))$, with smooth atlas $\{(U_\alpha=\varprojlim U^i_\alpha,\psi_\alpha=\varprojlim\psi^i_\alpha)\}_{a\in I}$ of M, $\{(p^{-1}(U_\alpha),\Psi_\alpha)\}_{a\in I}$ trivializations of $\mathcal{F}^2(M)$ and $\{\omega_\alpha:=s^*_\alpha\omega\}_{a\in I}$ the corresponding local forms of ω obtained as pull-backs with respect to the natural local sections $\{s_\alpha\}$ of $\{\Psi_\alpha\}$.

Then a (unique) linear connection can be defined on T^2M by means of the Christoffel symbols

$$\Gamma_\alpha : \psi_\alpha(U_\alpha) \to \mathcal{L}(\mathbb{F} \times \mathbb{F}, \mathcal{L}(\mathbb{F}, \mathbb{F} \times \mathbb{F}))$$

with $([\Gamma_\alpha(y)](u))(v) = \omega_\alpha(\psi_\alpha^{-1}(y))(T_y\psi_\alpha^{-1}(v))(u)$, $(y, u, v) \in \psi_\alpha(U_\alpha) \times \mathbb{F} \times \mathbb{F} \times \mathbb{F}$.

In Fréchet bundles an arbitrary connection is not easy to handle, since Fréchet manifolds and bundles lack a general theory of solvability for linear differential equations. Christoffel symbols (in the case of vector bundles) or the local forms (in principal bundles) are affected in representing linear maps since continuous linear mappings of a Fréchet space do not remain in the same category.

Galanis [14, 15] solved the problem for connections that can be obtained as projective limits and Dodson, Galanis and Vassiliou [8] obtained the following theorem with a number of areas of application.

Theorem ([8])

Let ∇ be a linear connection of the second order tangent bundle $T^2M = \varprojlim T^2M^i$ that can be represented as a projective limit of linear connections ∇^i on the (Banach modelled) factors. Then ∇ corresponds to a connection form ω of $\mathcal{F}^2M$ obtained also as a projective limit.

Choice of connection

Dodson, Galanis and Vassiliou [9] studied the way in which the choice of connection influenced the structure of the second tangent bundle over Fréchet manifolds, since each connection determines one isomorphism of $T^2M \equiv TM \bigoplus TM$.
They defined the second order differential T^2g of a smooth map $g : M \to N$ between two manifolds M and N.

In contrast to the case of the first order differential Tg, the linearity of T^2g on the fibres $(T^2_xg : T^2_xM \to T^2_{g(x)}N,\ x \in M)$ is not always ensured but they proved a number of results.

Following Vassiliou [27], connections ∇_M and ∇_N are called *g-conjugate* (or *g-related*) if they commute with those of g :

$$Tg \circ \nabla_M = \nabla_N \circ T(Tg).$$

Locally, for every $(x, u) \in U_\alpha \times \mathbb{E}$, the differentiation gives

$$Tg(\phi_\alpha(x))(\Gamma^M_\alpha(\phi_\alpha(x))(u)(u)) =$$
$$\Gamma^N_\beta(g(\phi_\alpha(x)))(Tg(\phi_\alpha(x))(u))(Tg(\phi_\alpha(x))(u))$$
$$+T(Tg)((\phi_\alpha(x))(u,u).$$

For g-conjugate connections ∇_M and ∇_N the local expression of $T^2_x g$ reduces to

$$(\Psi_{\beta,g(x)} \circ T^2_x g \circ \Phi^{-1}_{a,x})(u,v) = (Tg(\phi_\alpha(x))(u), Tg(\phi_\alpha(x))(v)).$$

Theorem ([9])

Let T^2M, T^2N be the second order tangent bundles defined by the pairs (M, ∇_M), (N, ∇_N), and let $g : M \to N$ be a smooth map. For g-conjugate connections ∇_M and ∇_N the second order differential $T^2g : T^2M \to T^2N$ is a vector bundle morphism.

Theorem ([9])

Let ∇, ∇' be two linear connections on M. If g is a diffeomorphism of M such that ∇ and ∇' are g-conjugate, then the vector bundle structures on T^2M, induced by ∇ and ∇', are isomorphic.

Differential equations

The importance of Fréchet manifolds arises from their ubiquity as quotient spaces of bundle sections and hence as environments for differential equations on such spaces. This was addressed next with Aghasi et al. [1] which provided a new way of representing and solving a wide class of evolutionary equations on Fréchet manifolds of sections.

First they considered a Banach manifold M, and defined an *integral curve* of ξ as a smooth map $\theta : J \to M$, defined on an open interval J of $\mathbb{R}$, if it satisfies the condition

$$T_t^2\theta(\partial_t) = \xi(\theta(t)).$$

Here ∂_t is the second order tangent vector of $T_t^2\mathbb{R}$ induced by a curve $c : \mathbb{R} \to \mathbb{R}$ with $c'(0) = 1$, $c''(0) = 1$.

If M is simply a Banach space $\mathbb{E}$ with differential structure induced by the global chart $(\mathbb{E}, id_{\mathbb{E}})$, then the generalization is clear since the above condition reduces to the second derivative of θ:

$$T_t^2\theta(\partial_t) = \theta''(t) = D^2\theta(t)(1,1).$$

Then we proved several theorems [1].

Theorem ([1])

Let ξ be a second order vector field on a manifold M modeled on Banach space $\mathbb{E}$. Then, the existence of an integral curve θ of ξ is equivalent to the solution of a system of second order differential equations on $\mathbb{E}$.

Of course, these second order differential equations depend not only on the choice of the second order vector field but also the choice of the linear connection that underpins the vector bundle structure.

In the case of a Banach manifold that is a Lie group, $M = (G, \gamma)$,

Theorem ([1])

Let v be any vector of the second order tangent space of G over the unitary element. Then, a corresponding left invariant second order vector field ξ of G may be constructed. Also, every monoparametric subgroup $\beta : \mathbb{R} \to G$ is an integral curve of the second order left invariant vector field ξ^2 of G that corresponds to $\ddot{\beta}(0)$.

Extending this to a Fréchet manifold M that is the projective limit of Banach manifolds from [6], yielded the result:

Theorem ([1])

Every second order vector field ξ on M obtained as projective limit of second order vector fields $\{\xi^i \text{ on } M^i\}_{i \in \mathbb{N}}$ admits locally a unique integral curve θ satisfying an initial condition of the form $\theta(0) = x$ and $T_t\theta(\partial_t) = y$, $x \in M$, $y \in T_{\theta(t)}M$, provided that the components ξ^i admit also integral curves of second order.

Ricci Flow on the manifold of Riemannian metrics

Ghahremani-Gol and Razavi [18] used the projective limit of Banach manifolds to represent the infinite dimensional space of Riemannian metrics on a compact manifold.

By this means they studied the parabolic partial differential equation for the Ricci Flow and its integral curves.

They found short-time solutions that are locally unique and in particular showed that a Ricci flow curve starting from an Einstein metric is not a geodesic.

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