A note on quantum chaology and gamma approximations to eigenvalue spacings for infinite random matrices*

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Abstract. Quantum counterparts of certain classical systems exhibit chaotic spectral statistics of their energy levels; eigenvalues of infinite random matrices model irregular spectra. Eigenvalue spacings for the Gaussian orthogonal ensemble (GOE) of infinite random real symmetric matrices admit a gamma distribution approximation, as do the hermitian unitary (GUE) and quaternionic symplectic (GSE) cases. Then chaotic and non chaotic cases fit in the information geometric framework of the manifold of gamma distributions, which has been the subject of recent work on neighbourhoods of randomness for general stochastic systems.
Keywords: Random matrices, GOE, GUE, GSE, quantum chaotic, eigenvalue spacing, statistics, gamma distribution, randomness, information geometry.

1 Introduction

Berry introduced the term quantum chaology in his 1987 Bakerian Lecture [6] as the study of semiclassical but non-classical behaviour of systems whose classical motion exhibits chaos. He illustrated it with the statistics of energy levels, following his earlier work with Tabor [7] and related developments from the study of a range of systems. In the regular spectrum of a bound system with $n \geq 2$ degrees of freedom and n constants of motion, the energy levels are labelled by n quantum numbers, but the quantum numbers of nearby energy levels may be very different. In the case of an irregular spectrum, such as for an ergodic system where only energy is conserved, we cannot use quantum number labelling. This prompted the use of energy level spacing distributions to allow comparisons among different spectra [7]. It was known, from the work of Porter [16], that the spacings between energy levels of complex nuclei and atoms with n large are modelled by the spacings of eigenvalues of random matrices and that the Wigner distribution [19] gives a very good fit. It turns out that the spacing distributions for generic regular systems are negative exponential, that is random; but for irregular systems the distributions are skew and unimodal, at the scale of the mean spacing.

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Mehta [14] provides a detailed discussion of the numerical experiments and functional approximations to the energy level spacing statistics, Alt et al [1] compare eigenvalues from numerical analysis and from microwave resonator experiments. Forrester's online book [12] gives a wealth of analytic detail on the mathematics and physics of eigenvalues of infinite random matrices for the three ensembles of particular interest: Gaussian orthogonal (GOE), unitary (GUE) and symplectic (GSE), being the real, complex and quaternionic cases, respectively. The review by Deift [9] illustrates how random matrix theory has significant links to a wide range of mathematical problems in the theory of functions as well as to mathematical physics. The matrices in these ensembles are respectively invariant under the appropriate orthogonal, unitary and symmetric transformation groups, and moreover in each case the joint density function of all independent elements is controlled by the trace of the matrices and is of form [12]

$$p(X) = A_n \, e^{-\frac{1}{2} Tr X^2} \tag{1}$$

where A_n is a normalizing factor. Barndorff-Nielsen et al [5] give some background mathematical statistics on the more general problem of quantum information and quantum statistical inference, including reference to random matrices.

Here we show that gamma distributions provide approximations to eigenvalue spacing distributions for the GOE distribution comparable to the Wigner distribution at the scale of the mean and for the GUE and GSE distributions, except near the origin. That may be useful in the study of irregular spectra of more general real systems and their perturbations because the gamma distribution has a well-understood and tractable information geometry, Arwini and Dodson [3,4,10], as well as the following important uniqueness property:

Theorem 1 (Hwang and Hu [13]). *For independent positive random variables with a common probability density function f, having independence of the sample mean and the sample coefficient of variation is equivalent to f being the gamma distribution.*

The non-chaotic case has an exponential distribution of spacings between energy levels. Now, the sum of n independent identical exponential random variables follows a gamma distribution and the sum of n independent identical gamma random variables follows a gamma distribution; moreover, the product of gamma distributions is well-approximated by a gamma distribution. Information geometry provides a distinguished information theoretic metric on the space of distributions and so allows comparison of trajectories through differing states during perturbations of parameters.

Monte Carlo methods were used by Caër et al. [8] who established the best fit of GOE, GUE and GSE unit mean distributions, for spacing $s > 0$, using the generalized gamma density which we can put in the form

$$g(s;\beta,\omega) = a(\beta,\omega)\, s^{\beta}\, e^{-b(\beta,\omega)s^{\omega}} \quad \text{for } \beta,\omega > 0 \tag{2}$$

$$\text{where}\quad a(\beta,\omega) = \frac{\omega\,[\Gamma((2+\beta)/\omega)]^{\beta+1}}{[\Gamma((1+\beta)/\omega)]^{\beta+2}} \quad\text{and}\quad b(\beta,\omega) = \left[\frac{\Gamma((2+\beta)/\omega)}{\Gamma((1+\beta)/\omega)}\right]^{\omega}.$$

Then the best fits of (2) had the parameter values shown in Table 1 and were accurate to within $\sim 0.1\%$ of the true distributions from Forrester [12]. Observe that the exponential distribution is recovered by the choice $g(s;0,1) = e^{-s}$. These distributions are shown in Figure 1 along with corresponding fits of the gamma distribution. More details of the study reported here and of the differential geometry of manifolds of probability density functions in application to near-random and other scenarios will be found in the forthcoming book Arwini and Dodson [3].

Ensemble	β	ω	Variance
GOE	1	1.886	0.2856
GUE	2	1.973	0.1868
GSE	4	2.007	0.1100

Table 1. *The best fit from Caër et al. [8] of GOE, GUE and GSE unit mean distributions, for eigenvalue spacing $s > 0$, using the generalized gamma density.*

2 Eigenvalues of Random Matrices

The mean spacing between eigenvalues of infinite symmetric real random matrices, the real Gaussian Orthogonal Ensemble (GOE), is bounded and therefore it is convenient to normalize the distribution to have unit mean; the same is true for the complex unitary (GUE) and quaternionic symplectic (GSE) cases. Wigner [17–19] had already surmised that the cumulative probability distribution function at the scale of the mean spacing should be of the form:

$$W(s) = 1 - e^{-\frac{\pi s^2}{4}} \tag{3}$$

which has unit mean and variance $\frac{4-\pi}{\pi} \approx 0.273$ with probability density function

$$w(s) = \frac{\pi}{2} s\, e^{-\frac{\pi s^2}{4}}. \tag{4}$$

Wigner's surmise gave a good fit with numerical computation of the true GOE distribution, Mehta [14] Appendix A.15, and with a variety of observed data from atomic and nuclear systems [19,7,6,14].

From Mehta [14] p 171, we have bounds on the cumulative probability distribution function P for the spacings between eigenvalues of infinite symmetric real random matrices:

$$L(s) = 1 - e^{-\frac{1}{16}\pi^2 s^2} \leq \; P(s) \; \leq U(s) = 1 - e^{-\frac{1}{16}\pi^2 s^2}\left(1 - \frac{\pi^2 s^2}{48}\right). \quad (5)$$

Here the lower bound L has mean $\frac{2}{\sqrt{\pi}} \approx 1.13$ and variance $\frac{4(4-\pi)}{\pi^2} \approx 0.348$, and the upper bound U has mean $\frac{5}{3\sqrt{5}} \approx 0.940$ and variance $\frac{96-25\pi}{9\pi^2} \approx 0.197$. The probability densities for the bounds (6) are, respectively,

$$l(s) = \frac{\pi s}{2} e^{-\frac{\pi s^2}{4}}, \quad u(s) = \frac{\pi^2 s(64 - \pi^2 s^2)}{384} e^{-\frac{1}{16}\pi^2 s^2}. \quad (6)$$

Probability density functions for gamma distributions with dispersion parameter $\kappa > 0$ and mean $\kappa/\nu > 0$ for positive random variable s is given by

$$f(s;\nu,\kappa) = \nu^\kappa \frac{s^{\kappa-1}}{\Gamma(\kappa)} e^{-s\nu}, \quad \text{for } \nu,\kappa > 0 \quad (7)$$

with variance $\frac{\kappa}{\nu^2}$. Then the subset having unit mean is given by

$$f(s;\kappa,\kappa) = \kappa^\kappa \frac{s^{\kappa-1}}{\Gamma(\kappa)} e^{-s\kappa}, \quad \text{for } \kappa > 0 \quad (8)$$

with variance $\frac{1}{\kappa}$. These parameters ν, κ are called natural parameters because they admit presentation of the family (7) as an exponential family, Amari and Nagaoka [2], and thereby provide an associated natural affine immersion in $\mathbb{R}^3$, Dodson and Matsuzoe [11]

$$h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^3 : \begin{pmatrix} \nu \\ \kappa \end{pmatrix} \mapsto \begin{pmatrix} \nu \\ \kappa \\ \log \Gamma(\kappa) - \kappa \log \nu \end{pmatrix}. \quad (9)$$

This affine immersion was used by Arwini and Dodson [3,4] to present tubular neighbourhoods of the 1-dimensional subspace consisting of exponential distributions ($\kappa = 1$), so giving neighbourhoods of random processes. The maximum entropy case has $\kappa = 1$ and corresponds to an underlying Poisson random event process and so models spacings in the spectra for non-chaotic systems; for $\kappa > 1$ the distributions are skew unimodular. The unit mean gamma distribution fit to the true GOE distribution from Mehta [14] has variance ≈ 0.379 and hence $\kappa \approx 2.42$.

In fact, κ is a geodesic coordinate in the Riemannian 2-manifold of gamma distributions with Fisher information metric; arc length along this coordinate from $\kappa = a$ to $\kappa = b$ is given by

$$\left| \int_a^b \sqrt{\frac{d^2 \log(\Gamma(\kappa))}{d\kappa^2} - \frac{1}{\kappa}}\, d\kappa \right|. \quad (10)$$

See Arwini and Dodson [3,4] for more details and related properties; also there can be found information geometry in neighbourhoods of uniform processes and neighbourhoods of independence for bivariate processes.

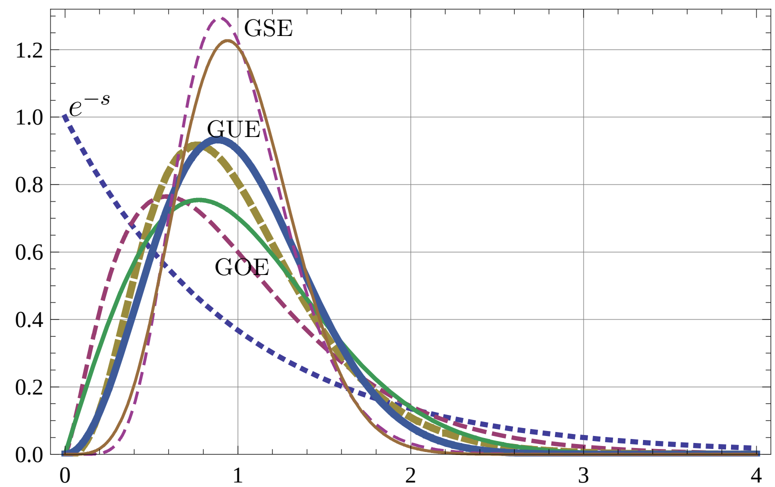

Fig. 1. *Probability density functions for the unit mean gamma distributions (dashed) and generalized gamma distribution (solid) fits to the true variances for left to right the GOE , GUE and GSE cases. The two types coincide in the random exponential case, e^{-s}, shown dotted.*

3 Deviations

Near the origin, the GOE and Wigner distributions are linear; the unitary ensemble (GUE) $\sim s^2$ and the symplectic ensemble (GSM) $\sim s^4$. At unit mean the gamma density behaves like $s^{\kappa-1}$ near the origin, so linearity would require $\kappa = 2$ with variance $\frac{1}{\kappa} = \frac{1}{2}$; the GOE fitted gamma distribution has $\kappa \approx 2.42$ and hence variance ≈ 0.379. Figure 1 shows the probability density functions for the unit mean gamma distributions (dashed) and generalized gamma distribution (solid) fits to the true variances for left to right the GOE, GUE and GSE cases; the two types coincide in the random case, which is exponential, e^{-s}, shown dotted. Figure 2 shows gamma best fits to the true GOE, GUE and GSE cases, as points on the affine immersion in $\mathbb{R}^3$ of the 2-manifold of gamma distributions, cf. [3,11]. The information metric provides information distances on the gamma manifold and so could be used for comparison of real data on eigenvalue spacings if fitted to gamma distributions;

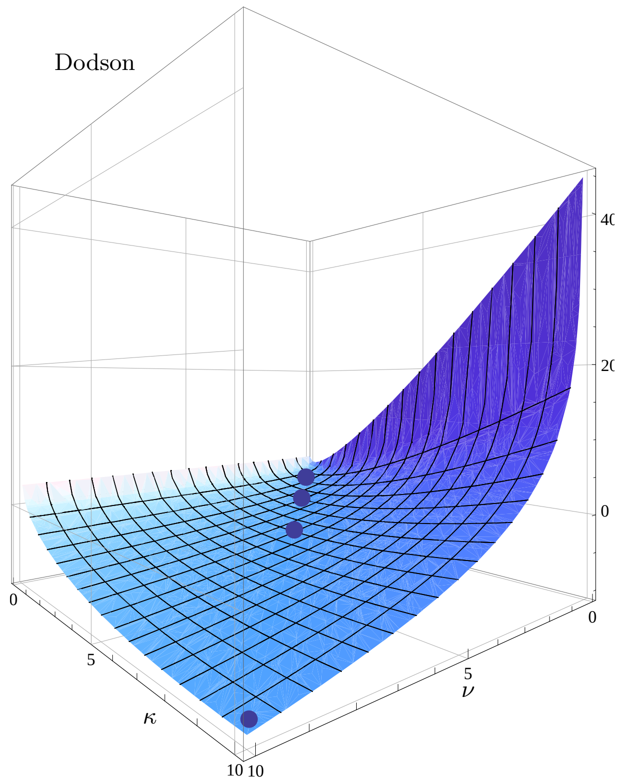

Fig. 2. *The unit mean gamma distributions corresponding to the random (non-chaotic) case,* $\kappa = \nu = 1$ *and those with exponent* $\kappa = \nu = 2.420,\ 4.247,\ 9.606$ *for the best fits to the true variances of the spacing distributions for the GOE, GUE and GSE cases, as points on the affine immersion in* $\mathbb{R}^3$ *of the* 2*-manifold of gamma distributions.*

that may allow identification of qualitative properties and represent trajectories during structural changes of systems. Figure 3 shows with unit mean the probability distribution for spacings among the first 2,001,052 zeros from the tabulation of Odlyzko [15] (large points), that for the true GUE distribution from the tabulation of Mehta [14] Appendix A.15 (medium points) and the gamma fit to the true GUE (small points), which has $\kappa \approx 4.247$. The grand mean spacing between zeros from the data was ≈ 0.566, the coefficient of variation ≈ 0.422 and variance ≈ 0.178.

The effect of location on the statistical data for spacings in the first ten consecutive blocks of 200,000 zeros of the Riemann zeta function normalized with unit grand mean and the effect of sample size were investigated in Arwini and Dodson [3]. For gamma distributions we expect the coefficient of variation to be independent of sample size and location, by Theorem 1 and this was approximately the case.

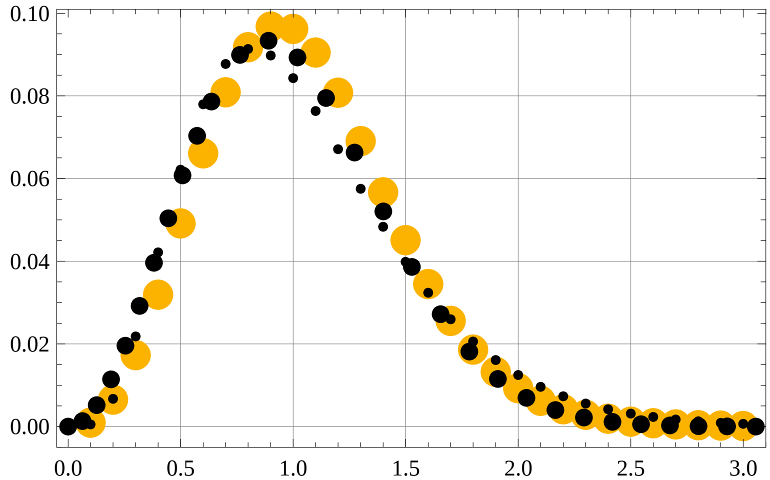

Fig. 3. *Probability plot with unit mean for the spacings between the first 2,001,052 zeros of the Riemann zeta function from the tabulation of Odlyzko [15] (large points), that for the true GUE distribution from the tabulation of Mehta [14] Appendix A.15 (medium points) and the gamma fit to the true GUE (small points).*

4 Conclusion

The gamma distribution provides approximations to the true distributions for the spacings between eigenvalues of infinite random matrices for the GOE, GUE and the GSE cases. However, it is clear that gamma distributions do not precisely model the analytic systems discussed here, and do not give correct asymptotic behaviour at the origin, as is evident from the results of Caër et al. [8] who obtained excellent approximations for GOE, GUE and GSE distributions using the generalized gamma distribution (2). The differences may be seen in Figure 1 which shows the unit mean distributions for gamma (dashed) and generalized gamma (solid) fits to the true variances for the Poisson, GOE, GUE and GSE ensembles. Unfortunately, the generalized gamma distributions do not have a tractable information geometry and so some features of the gamma distribution approximations may be useful in studies of qualitative generic properties in applications to data from real systems. It would be interesting to investigate the extent to which data from real atomic and nuclear systems has generally the qualitative property that the sample coefficient of variation is independent of the mean. That, by Theorem 1, is an information-theoretic distinguishing property of the gamma distribution.

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