Remark. Let a compact set X be either T^n or S^n. Let B be an algebra of complex-valued continuous functions on X, such that $\log|B^{-1}| = \log|A^{-1}|$, where A is, respectively, either $A(T^n)$ or $A(S^n)$; where under $\log|B^{-1}|$ we understand the following set: $\{\varphi(z_1, \ldots, z_n) \in C(X)$: there exists an invertible $f(z_1, \ldots, z_n) \in B$ such that $\log|f| = \varphi(z_1, \ldots, z_n)\}$. Then either B = A or $B = \bar{A}$.

Indeed, under the above assumptions we have $\operatorname{Re} B \subset \log|A^{-1}|$ and $\operatorname{Re} A \subset \log|B^{-1}|$. By the corollary to a lemma from [7] we can conclude that $\operatorname{Re} B \subset \operatorname{Re} A$ and $\operatorname{Re} A \subset \operatorname{Re} B$. Consequently, Re B = Re A and then B = A or $B = \bar{A}$.

LITERATURE CITED

1. J. M. F. O'Connell, "Real parts of uniform algebras," Pacific J. Math., 46, 235-247 (1973).
2. W. P. Nowinger, "Real parts of uniform algebras on the circle," Pacific J. Math., 57, No. 1, 259-264 (1975).
3. J. Hamelin, Uniform Algebras [Russian translation], Mir, Moscow (1973).
4. M. I. Zaslavskaya, "On some representations of uniform algebras on the unit circle and polytorus," Uch. Zametki EGU (1989).
5. U. Rudin, Theory of Functions on a Disc [Russian translation], Mir, Moscow (1973).
6. U. Rudin, Theory of Functions on the Unit Ball in C^n [Russian translation], Mir, Moscow (1984).
7. B. T. Batikyan, "Logarithms of moduli of invertible elements in a Banach algebra," Mat. Zametki, 23, No. 3, 373-376 (1978).

TANGENT AND FRAME BUNDLE HARMONIC LIFTS

C. T. J. Dodson*† and M. E. Vazquez-Abal‡

UDC 517

For a map f:(M, g) → (N, h) between Riemannian manifolds we study harmonicity in the induced tangent and frame bundle diagram

$$\begin{array}{ccccc} (FM, Fg) & \to & (M, g) & \leftarrow & (TM, Tg) \\ \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Tf} \\ (FN, Fh) & \to & (N, h) & \leftarrow & (TN, Th). \end{array}$$

with respect to the diagonal lifts of base metrics; here Ff is well-defined if f is a local diffeomorphism. In each case the bundle projection is harmonic and has fibers which are totally geodesic and hence minimal submanifolds, so we have harmonic fibrations. We prove that, when Ff is defined, it is totally geodesic if and only if f is totally geodesic, and if f is a local diffeomorphism of flat manifolds then Ff is harmonic whenever f is harmonic. This extends to the frame bundle for a number of results of Sanini for the tangent bundle. Consideration is given also to another Riemannian structure induced on a frame bundle by a linear connection on the base manifold; it gives rise also to a harmonic fibration and for this some stability properties are known concerning incompleteness.

Introduction

On the tangent bundle TM to a Riemannian m-manifold (M, g) Sasaki [18] introduced a natural Riemannian structure Tg, the diagonal lift of g. Mok [15] devised in a similar way a Riemannian structure Fg on FM, the principal $G\ell(m)$-bundle of linear frames on M. In this paper we consider harmonicity and total geodesicity in the bundle diagrams induced by a local diffeomorphism f from (M, g) to a Riemannian n-manifold (N, h):

---

*Department of Mathematics, University of Lancaster, England.
†Present address: Department of Mathematics, University of Toronto, Canada M5S 1A4.
‡Department of Geometry and Topology, University of Santiago de Compostela, Spain.

Published in Matematicheskie Zametki, Vol. 50, No. 3, pp. 27-37, September, 1991. Original article submitted May 25, 1988.

$$\begin{array}{ccccc}(FM, Fg) & \rightarrow & (M, g) & \leftarrow & (TM, Tg) \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Tf} \\ (FN, Fh) & \rightarrow & (N, h) & \leftarrow & (TN, Th).\end{array}$$

In each case the induced metrics make orthogonal the horizontal and vertical distributions induced by the Levi-Civita connections ∇^g and ∇^h and we have harmonic Riemannian submersions, with geodesics preserved under projection and horizontal lifting. For a Riemannian submersion, completeness of the total space implies completeness of its fibers and of the base space (cf. Hermann [10], O'Neill [16]). Kowalski [13] studied the curvature of Tg and established the result

$$\nabla^{Tg} R^{Tg} = 0 \Rightarrow R^g = 0 \Rightarrow R^{Tg} = 0.$$

Thus, flatness of (M, g) lifts to flatness of (TM, Tg) and nonflatness of (M, g) lifts to lack of local symmetry in (TM, Tg). Moreover, from Fernandez and de Leon [8], we deduce that nonflatness of (M, g) forces nonconstancy of the scalar curvature of (TM, Tg), and unboundedness of its sectional curvature follows from Aso [1].

From Mok [15] we have the analogous situation

$$\nabla^{Fg} R^{Fg} = 0 \Rightarrow R^g = 0 \Rightarrow R^{Fg} = 0.$$

Cordero and de Leon [4] proved that (FM, Fg) is flat if its sectional curvature is bounded or if it has the same constant scalar curvature as (M, g), or if it is an Einstein manifold.

Another Riemannian structure is available on the frame bundle of any manifold with a linear connection; it also yields a harmonic Riemannian submersion onto the base and has totally geodesic fibers. Recent studies of this space by Canarutto and Dodson [3] and Del Riego and Dodson [5] may be relevant to questions of stability of harmonicity.

The coordinate expression for the second fundamental form of $f:(M, g) \to (N, h)$ is given by

$$[\nabla^g \, df]^{\gamma}_{ij} = \partial^2_{ij} f^{\gamma} - {}^g\Gamma^k_{ij} \partial_k f^{\gamma} + {}^h\Gamma^{\gamma}_{\alpha\beta} \partial_i f^{\alpha} \partial_j f^{\beta},$$

and its trace $T(f)$ appears locally as

$$\tau(f)^{\gamma} = g^{ij} (\nabla^g \, df)^{\gamma}_{ij}.$$

The map f is called harmonic if $\tau(f) = 0$ (cf. [7]).

The Sasaki metric Tg has components diag(g_{ij}, g_{ij}) with respect to the horizontal-vertical splitting induced by ∇^g. The second fundamental form of $\pi_{TM}: TM \to M$ has coordinate expression

$$(\nabla^{Tg} \, d\pi_{TM})^k = \begin{bmatrix} 0 & -\frac{1}{2} \, {}^g R^k_{ilj} y^l \\ -\frac{1}{2} \, {}^g R^k_{jli} y^l & 0 \end{bmatrix},$$

where ${}^g R^k_{i\ell j}$ denotes components of R^g, the curvature of ∇^g. Clearly $\tau(\pi_{TM}) = 0$ and π_{TM} is totally geodesic if and only if (M, g) is flat. When M is compact, the energy of f is defined to be the integral of $(1/2)|df|^2$; then f is known to be harmonic if and only if it is an extremal for this energy.

Tangent Bundle

Sanini [12] established the following results.

(a) Tf is totally geodesic $\Leftrightarrow f$ is totally geodesic.

If f is harmonic, then (b) Tf is harmonic $\Leftrightarrow$

$$\begin{cases} \operatorname{div}(\nabla^h df) = 0 \\ R^h(\nabla^h df\,(X, e_i), dfX) df\, c_i = 0 \\ \text{for all vector fields } X \text{ on } M \text{ and orthonormal frames } (c_i), \\ R^h \text{ being the curvature of } \nabla^h. \end{cases}$$

Hence, if f is a map between flat manifolds, then f harmonic $\Rightarrow Tf$ harmonic.

(c) If M is compact, then Tf harmonic $\Rightarrow f$ totally geodesic. □

Now consider the tangent bundle diagram induced by f

$$\begin{array}{ccc} (TM, Tg) & \xrightarrow{Tf} & (TN, Th) \\ \pi_{TM} \downarrow & & \downarrow \pi_{TN} \\ (M, g) & \xrightarrow{f} & (N, h). \end{array}$$

By direct computation we find that π_{TM} is a harmonic Riemannian submersion and so by Smith [13] we have the following:

THEOREM 1. In the diagram, the diagonal map

$$\pi_{TN} \circ Tf = f \circ \pi_{TM}$$

is harmonic if and only if f is harmonic. □

COROLLARY 1. Suppose that (N, h) is **R** and (TM, Tg) is complete with nonnegative sectional curvature. Then, for compact (M, g) the real function

$$f \circ \pi_{TM} : TM \to \mathbf{R},$$

is constant if it has bounded energy.

Proof. This is a special case of a theorem of Greene and Wu [9]. □

COROLLARY 2. Suppose that (N, h) is compact with nonpositive sectional curvature and (TM, Tg) is complete with nonnegative Ricci curvature. Then, for compact (M, g), the map

$$f \circ \pi_{TM} : TM \to N,$$

is constant if it has bounded energy.

Proof. This follows from a result of Schoen and Yau [19]. □

COROLLARY 3. π_{TN} is totally geodesic if and only if the horizontal distribution of TN is integrable; then $f \circ \pi_{TM}$ is harmonic if Tf is harmonic.

Proof. This follows from a result of Vilms [23] because π_{TM} is a Riemannian submersion with totally geodesic fibers. □

COROLLARY 4. In (TN, Th), if either the scalar curvature is constant or the sectional curvature is bounded, then π_{TN} is totally geodesic, and so $f \circ \pi_{TM}$ is harmonic if Tf is harmonic.

Proof. Each property is sufficient to ensure flatness of (N, h) by theorems of Fernandez and de Leon [8] and Aso [1], respectively; then π_{TN} is totally geodesic and we apply Corollary 3. □

Note that, since π_{TM} is a Riemannian submersion, if (TM, Tg) is complete, then so is (M, g).

Frame Bundle

The metric Fg introduced by Mok [15] on the frame bundle FM of a Riemannian m-manifold (M, g) resembles that of Sasaki [18] for the tangent bundle. For it is a diagonal lift making orthogonal the horizontal and vertical distributions induced by the Levi-Civita connection ∇^g on the base and it makes the projection π_{FM} a harmonic Riemannian submersion, actually a harmonic fibration because its fibers are minimal submanifolds.

We follow Mok [15] and express the metric Fg with respect to the adapted coframe for TFM

$$(dx^i, {}^g\Gamma^h_{ij} X^j_\alpha \, dx^j + dX^h_\alpha) = (dx^i, \delta X^h_\alpha),$$

where coordinates (x^i) on M induce coordinates (x^i, X^i_α) on FM. Then we obtain, locally,

$$Fg = g_{ij} dx^i \otimes dx^j + \delta_{\alpha\beta} g_{ij} \delta X^i_\alpha \otimes \delta X^j_\beta.$$

It follows that the second fundamental form of π_{TM} has coordinate expression

$$(\nabla^{Fg} \, d\pi_{FM})^k = \begin{bmatrix} 0 & -\frac{1}{2} {}^g R^k_{ilj} x^l_\alpha \\ -\frac{1}{2} {}^g R^k_{jli} x^l_\alpha & 0 \end{bmatrix}$$

and hence $\tau(\pi_{FM}) = 0$; again, π_{FM} is totally geodesic if and only if (M, g) is flat.

A direct computation establishes that π_{FM} is a harmonic Riemannian submersion with totally geodesic fibers. Less directly, each fiber is an autoparallel submanifold by Mok [15] and therefore totally geodesic and minimal [12]. Observe that any fibered manifold map f: M ← N, in other words, a surjective submersion, induces a map from FM to a quotient bundle of FN. However, a well-defined map Ff:FM → FN exists when f is a local diffeomorphism. A further application of Smith [21] yields the following.

THEOREM 2. In the diagram

$$\begin{array}{ccc} (FM, Fg) & \xrightarrow{Ff} & (FN, Fh) \\ \pi_{FM} \downarrow & & \downarrow \pi_{FN} \\ (M, g) & \xrightarrow{f} & (N, h). \end{array}$$

let f be a local diffeomorphism; then the diagonal map

$$\pi_{FN} \circ Tf = f \circ \pi_{FM}$$

is harmonic if and only if f is harmonic. □

COROLLARY 1. When Ff is harmonic, then any one of the following conditions is sufficient to ensure harmonicity of f:

(a) (N, h) or (FN, Fh) is flat;

(b) (FN, Fh) is an Einstein manifold;

(c) (FN, Fh) has bounded sectional curvature.

Proof. Suppose that Ff is harmonic. If (N, h) is flat, then $\pi_{FN} \circ$ Ff is harmonic; so by the theorem, f is harmonic; this establishes (a) since Mok [15] showed that (FN, Fh) is flat if and only if (N, h) is flat.

For (b), we observe that Cordero and de Leon [4] proved that (FN, Fh) is an Einstein manifold only if (N, h) is flat, so the result follows from (a).

For (c), if the sectional curvature of (FN, Fh) is bounded, then (FN, Fh) is flat [4] and we apply (a). □

As before, completeness of (M, g) is implied by completeness of (FM, Fg). For compact (M, g) the energy integral is defined (cf. [7]) and we have the following two applications.

COROLLARY 2 (again using Greene and Wu [9]). Suppose that (N, h) is **R** and (FM, Fg) is complete with nonnegative sectional curvature. Then,

$$f \circ \pi_{FM} : FM \to \mathbf{R},$$

is constant if it has bounded energy. □

COROLLARY 3 (again using Schoen and Yau [19]). Suppose that (N, h) is compact with nonpositive sectional curvature and (FM, Fg) is complete with nonnegative Ricci curvature. Then

$$f \circ \pi_{FM}: FM \to N,$$

is constant if it has bounded energy. □

In the presence of the Riemannian structure g on M, the frame bundle FM is reducible to the orthonormal bundle OM with structure group O(m). Mok [15] has studied the geometry of OM as a Riemannian submanifold of (FM, Fg). He showed that the complete (i.e., natural) lift to FM of an infinitesimal isometry (i.e., Killing vector field) on (M, g) induces an infinitesimal isometry on OM.

As might be expected, the lifts Tf and Ff for a local diffeomorphism f:(M, g) → (N, h) are related. Moreover the Sasaki and Mok lifts of the Riemannian structures have common features and we obtain the following result for harmonic maps, extending Sanini's [17] theorem to the frame bundle.

THEOREM 3. Let f:(M, g) → (N, h) be a local diffeomorphism; then:

(a) Ff:(FM, Fg) → (FN, Fh) is totally geodesic if and only if f is totally geodesic;

(b) Ff is harmonic if and only if Tf is harmonic;

(c) if (M, g) and (N, h) are flat, then f harmonic $\Rightarrow Ff$ harmonic;

(d) if M is compact, then Ff harmonic $\Rightarrow f$ totally geodesic.

Proof. We show that Ff is totally geodesic if and only if Tf is totally geodesic and then use Sanini [17].

(a) The second fundamental form for Ff has components

$$[[\tilde{\nabla}\, dFf]^k, [\tilde{\nabla}\, dFf]^k_\gamma] \in \mathbf{R}^{(m+m^2)\times(m+m^2)} \times \mathbf{R}^{(n+n^2)\times(n+n^2)},$$

where $\tilde{\nabla}$ is the Levi-Civita connection of Fg, and m = n. These have the following appearance

$$(\tilde{\nabla}\, dFf)^k = \begin{bmatrix} \sum_\alpha (A^k_{ji})_{ab}\, X^a_\alpha X^b_\beta + (\nabla\, df)^k_{ji}\, (C^k_{ji})_a\, X^a_\alpha \\ ((B^k_{ji})_a\, X^a_\alpha)^t & 0 \end{bmatrix},$$

$$(\tilde{\nabla}\, dFf)^k_\gamma = \begin{bmatrix} (D^k_{ji})_a\, X^a_\gamma & (\nabla\, df)^k_{ji}\, \delta^\gamma_\alpha + (G^k_{ji})_{ab}\, X^a_\alpha X^b_\gamma \\ ((E^k_{ji})\, \delta^\gamma_\alpha + (F^k_{ji})_{ab}\, X^a_\alpha X^b_\gamma)^t & 0 \end{bmatrix}$$

for certain arrays, A, B, C, D, E, F, G which depend on f and the curvatures of (M, g) and (N, h).

Similarly, for Tf we have the components

$$((\bar{\nabla}\, dTf)^k, (\bar{\nabla}\, dTf)^{k+n}) \in \mathbf{R}^{2n\times 2n} \times \mathbf{R}^{2n\times 2n},$$

where $\bar{\nabla}$ is the Levi-Civita connection of Tg, and they are summarized by

$$(\bar{\nabla}\, dTf)^k = \begin{bmatrix} (A^k_{ji})_{ab}\, Y^a Y^b + (\nabla\, df)^k_{ji}\, (C^k_{ji})_a\, Y^a \\ (B^k_{ji})_a\, Y^a & 0 \end{bmatrix},$$

$$(\bar{\nabla}\, dTf)^{k+n} = \begin{bmatrix} (D^k_{ji})_a\, Y^a & (\nabla\, df)^k_{ji} + (G^k_{ji})_{ab}\, Y^a Y^b \\ (E^k_{ji}) + (F^k_{ji})_{ab}\, Y^a Y^b & 0 \end{bmatrix}.$$

Evidently, the coordinates X^a_α in FM run through all nonsingular (n × n) matrices and the coordinates Y^a in TM run through $\mathbf{R}^n$. It follows from the above expressions that $\tilde{\nabla} dFf = 0$ if and only if $\bar{\nabla} dTf = 0$.

(b) This follows from inspection of the above components since Fg and Tg are diagonal lifts of g.

(c) If (M, g) and (N, h) are flat, then harmonicity of f implies harmonicity of Tf and hence also of Ff when it is defined.

(d) This follows from (b) and Sanini [17]. □

Another quite natural Riemannian structure is available on the frame bundle to any manifold with a linear connection. This seems to have been used first, independently, by Marathe [14] and Schmidt [20] (though O'Neill [16] studied the case for a Riemannian connection). For a detailed study of the induced geometry and its application to the study of spacetime singularities, see Dodson [6]. Let ∇ be a linear connection with connection form ω_∇ on M and denote by θ the canonical 1-form. Then an induced Riemannian structure, the connection metric, is given on FM by

$$g_\nabla = \theta \cdot \theta + \omega_\nabla \cdot \omega_\nabla,$$

where · denotes the standard inner product on $\mathbf{R}^m$ and $\mathbf{R}^{m^2}$. We can express the metric g_∇ with respect to the adapted coframe for TFM

$$(dx^i, {}^\nabla\Gamma^k_{ij} X^i_\alpha\, dx^i + dX^k_\alpha) = (dx^i, \delta X^k_\alpha),$$

where coordinates (x^j) on M induce coordinates (x^i, X^i_α) on FM and ${}^\nabla\Gamma^k_{ij}$ are the components of the connection ∇ on M. Then we obtain locally

$$g_\nabla = g_{\nabla_{ij}}\, dx^i \otimes dx^j + \delta_{\alpha\beta}\, g_{\nabla_{ij}}\, \delta X^i_\alpha \otimes \delta X^j_\beta,$$

where $g_{\nabla_{ij}} = \sum_\gamma Y^\gamma_i Y^\gamma_j$ and $Y^i_j = (X^i_j)^{-1}$.

In the present context we are interested in the situation where the linear connection ∇ coincides with the Levi-Civita connection ∇^g of a Riemannian structure g on M. Then the metric g_{∇^g} so induced is the connection metric of the metric connection.

We use $\hat{g}$ to denote the restriction to OM of the connection metric $g_{\nabla g}$. Note that if $Y = (x, b, X, B)$ is a tangent vector to OM at (x, b), then the splitting of TOM induced by ∇g yields

$$Y = Y^H \otimes Y^V = (x, b, X, -b\Gamma X) \otimes (x, b, 0, B + b\Gamma X)$$

in an obvious matrix notation, Γ representing the Christoffel symbols. Then

$$\theta(Y) = b^{-1}X \text{ in } \mathbf{R}^m$$

and

$$\omega_{\nabla_g}(Y) = (B + b\Gamma X)\, b^{-1} \text{ in } \mathbf{R}^{m^2}.$$

THEOREM 4. If (M, g) is a Riemannian m-manifold, then

(i) $\hat{g} = \theta\cdot\theta + \omega_{\nabla g}\cdot\omega_{\nabla g}$ is uniformly equivalent to every $\tilde{g} = \theta * \theta + \omega_{\nabla g} \otimes \omega_{\nabla g}$, where $*$, $\otimes$ are any other inner products on $\mathbf{R}^m$, $\mathbf{R}^{m^2}$.

(ii) O(m) acts uniformly continuously on (OM, $\hat{g}$).

(iii) (OM, $\hat{g}$) is complete if and only if (M, g) is complete.

(iv) If (OM, $\hat{g}$) is incomplete, then its completion $\overline{OM}$ quotients by O(m) yield a homeomorph of the completion $\overline{M}$ of (M, g).

(v) The bundle (OM, $\hat{g}$) $\xrightarrow{\pi_{OM}}$ (M, g) is a harmonic submersion but not a Riemannian submersion.

Proof. Parts (i)-(iv) are proved in [6], where results are given also on completion of associated bundles. Part (v) follows from a direct calculation. □

Observe that for this submersion but completeness and incompleteness lift from the base manifold.

O'Neill [16] has computed the sectional curvature of (OM, $\hat{g}$) and has given a number of useful formulas for general submersions.

THEOREM 5. Let $A:O(m) \to O(n)$ be an epimorphism, and suppose that $\Phi:OM \to ON$ is an A-equivariant orthonormal bundle morphism [i.e., $\Phi(\alpha\cdot y) = A(\alpha)\cdot\Phi(y)$] over Riemannian manifolds (M, g), (N, h). Then

(i) Trace $\nabla^{\hat{g}} d\Phi = \tau(\Phi)$ is equivariant.

(ii) Φ is harmonic if and only if it is an extremal of the energy with respect to all compactly supported equivariant variations.

Proof. It can be seen that the action of O(m) on OM is isometric, and then the result follows as a special case of a theorem of Smith [21]. □

See Eells and Lemaire [7, pp. 17-18] for further discussion and a summary of Smith's necessary and sufficient conditions for the induced map on quotients, $\overline{\Phi}:M \to N$, to be harmonic. In particular, we can deduce the following.

COROLLARY. Let $f:(M, g) \to (N, h)$ lift to an A-equivariant map $Ff:(OM, \hat{g}) \to (ON, \hat{h})$ which preserves horizontality and suppose that $A:O(m) \to O(n)$ is a Riemannian submersion. Then f is harmonic if and only if Ff is harmonic. □

We note that, given Φ as in the theorem and a connection in ON, it is always possible to induce a connection in OM such that Φ preserves horizontality (and curvature forms) [11].

THEOREM 6. Incompleteness of (FM, g_∇) is stable under variation of the connection ∇ on M.

Proof. This is formulated precisely and proved in Canarutto and Dodson [3]. □

In particular, it turns out that incompleteness of (FM, $g_{\nabla g}$) persists into the family of spaces (FM, $g'_{\nabla g'}$) induced by a 1-parameter family $\{g'\}$ of metrics conformal to g. Of curse, a sufficiently large conformal change can always complete an incomplete Riemannian manifold; but this is not true, for example, for geodesically incomplete Lorentzian manifolds (cf. Beem [2]). These theorems suggest one way to approach the study of stability of harmonicity under variation of the geometry. The basis of the method in Theorem 6 is the canonical universal connection on a space of connections (cf. [5]). The generality here may allow also the formulation of an appropriate generalization of the notion of harmonicity to maps between manifolds with connection. For example, each choice of linear connection ∇ on M induces a Riemannian isometric imbedding of (FM, g_∇) into the bundle of linear connections on M.

Acknowledgment

The authors wish to thank The British Council for support of M.E. Vazquez-Abal at Lancaster University during the summer of 1985 when the study reported in this paper was begun.

LITERATURE CITED

1. K. Aso, "Notes on some properties of the sectional curvature of the tangent bundle," Yokohama Math, J., 29, 1-5 (1980).
2. J. K. Beem, "Conformal changes and geodesic completeness," Commun. Math. Phys., 49, 179-186 (1976).
3. D. Canarutto and C. T. J. Dodson, "On the bundle of principal connections and the stability of b-incompleteness of manifolds," Math. Proc. Cambridge Philos. Soc., 98, 51-59 (1985).
4. L. Cordero and M. de Leon, "On the curvature of the induced Riemannian metric on the frame bundle of a Riemannian manifold," J. Math. Pures Appl., 65, 81-91 (1986).
5. L. Del Riego and C. T. J. Dodson, "Sprays, universality and stability," Math. Proc. Cambridge Philos. Soc., 103, 515-534 (1988).
6. C. T. J. Dodson, "Spacetime edge geometry," Int. J. Theor. Phys., 17, No. 6, 389-504 (1978).
7. J. Eells and L. Lemaire, "A report on harmonic maps," Bull. London Math. Soc., 10, 1-68 (1978) [cf. also "Selected topics in harmonic maps," Conference Board of the Mathematical Sciences, Regional Conference Series in Mathematics No. 50, American Mathematical Society (1983)].
8. M. Fernandez and M. de Leon, "Some properties of the holomorphic sectional curvature of the tangent bundle," Preprint, Fac. Matematicas, Univ. Santiago de Compostela.
9. R. E. Greene and H. H. Wu, "Integrals of subharmonic functions on manifolds of nonnegative curvature," Invent. Math., 27, 265-298 (1974).
10. R. Hermann, "A sufficient condition that a mapping of Riemannian manifolds be a fiber bundle," Proc. Am. Math. Soc., 11, 236-242 (1960).
11. S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. 1, Interscience, New York (1963).
12. S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. II, Interscience, New York (1969).
13. O. Kowalski, "Curvature of the induced Riemannian metric on the tangent bundle of a Riemannian manifold," J. Reine. Angew. Math., 250, 124-129 (1971).
14. K. B. Marathe, "A condition for paracompactness of a manifold," J. Diff. Geom., 7, 571-573 (1972).
15. K.-P. Mok, "On the differential geometry of frame bundles of Riemannian manifolds," J. Reine, Angew. Math., 302, 16-31 (1978) [cf. also "Complete lifts of tensor fields and connections to the frame bundle," Proc. London Math. Soc., 3, 38, 72-88 (1979)].
16. B. O'Neill, "The fundamental equations of a submersion," Mich. Math. J., 13, 456-469 (1966).
17. A. Sanini, "Applicazioni armoniche tra i fibrati tangenti di varieta riemanniane," Bollettino U.M.I., (6), 2-A, 55-63 (1983).
18. S. Sasaki, "On the differential geometry of tangent bundles of Riemannian manifolds," Tohoku Math. J., 10, 338-354 (1958).
19. R. Schoen and S. T. Yau, "Harmonic maps and the topology of stable hypersurfaces and manifolds of nonnegative Ricci curvature," Commun. Math. Helv., 51, 333-341 (1976).
20. B. G. Schmidt, "A new definition of singular points in general relativity," Gen. Relativ. Gravit., 1, 269-280 (1971).
21. R. T. Smith, Harmonic Mappings of Spheres, Thesis, University of Warwick (1972).
22. R. T. Smith, "Harmonic mappings of spheres," Am. J. Math., 97, No. 1, 364-385 (1975).
23. J. Vilms, "Totally geodesic maps," J. Diff. Geom., 4, 73-79 (1970).