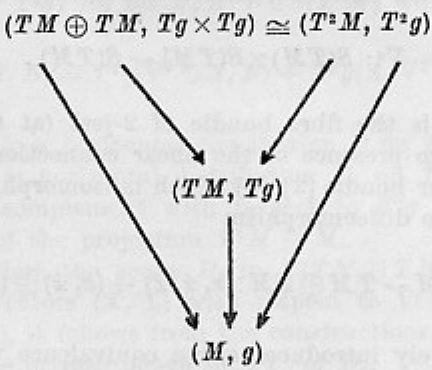
Bollettino U. M. I.
(7) 4-B (1990), 943-952

Harmonic Fibrations of the Tangent Bundle of Order Two.

C. T. J. DODSON - M. E. VÁZQUEZ-ABAL

Sunto. – *In un precedente lavoro sono state studiate le fibrazioni armoniche del fibrato tangenti e del fibrato dei riferimenti su una m-varietà riemanniana (M, g). Si considerano strutture corrispondenti per T^2M, il fibrato tangente del secondo ordine, che è costituito dai 2-jets di curve in M. La metrica riemanniana g su M induce attraverso la sua connessione una metrica riemanniana naturale T^2g su T^2M. Sfruttando la metrica Tg di Sasaki* ([8](*su TM si ha il seguente diagramma di fibrazioni armoniche.*



in cui la riga superiore è un'isometria. Si ottengono diversi risultati riguardanti la geometria di T^2M e si considerano anche l'armonicità delle mappe indotte fra due tali diagrammi da un mappa da (M, g) a una varietà riemanniana (N, h).

1. – Second order geometry.

Throughout we shall work in smooth categories of real manifolds with boundary and in finite dimensions. Let (M, g) be a Riemannian m-manifold, then its tangent bundle TM becomes a Riemannian $\mathbb{R}^m$-vector bundle with respect to g on its fibres and it has a unique torsion free linear connection ∇^g which is compatible with g. In a natural way, the Whitney sum $TM \oplus TM \subset TM \times TM$ is a Riemannian $\mathbb{R}^m \oplus \mathbb{R}^m$-vector bundle with respect to (the restriction of) the metric $g \times g$; its unique torsion free linear connection compatible with $g \times g$ is $\nabla^g \oplus \nabla^g$.

Now, TM admits also a Riemannian metric Tg, the Sasaki or diagonal lift of g; so (TM, Tg) is a Riemannian $2m$-manifold. The local components of Tg are diag (g_{ij}, g_{ij}) with respect to the tangent frame field

$$\{\partial_i - \dot{x}^j \Gamma^k{}_{ij} \dot{\partial}_k, \dot{\partial}_i\} = \{h_i, \dot{\partial}_i\}$$

at a point in TM with coordinates $(x^i, \dot{x}^i)$ and induced natural tangent frame $\{\partial_i, \dot{\partial}_i\}$. Here $\Gamma^k{}_{ij}$ denotes the Christoffel symbols of ∇^g and hence locally the $\{h_i\}$ span the ∇^g-horizontal distribution in TTM. On the tangent bundle, ∇^g is also a vector bundle connection; that is, a covariant derivation of sections:

$$\nabla^g : S(TM) \times S(TM) \to S(TM) .$$

Next, T^2M is the fibre bundle of 2-jets (at $0 \in \mathbb{R}$) of curves in M and in the presence of the linear connection ∇^g it becomes a $\mathbb{R}^m \oplus \mathbb{R}^m$-vector bundle [2], [6] which is isomorphic to $TM \oplus TM$ by means of the diffeomorphism

$$\Gamma_g : T^2M \to TM \oplus TM : (x, \dot{x}, \ddot{x}) \to (x, \dot{x}) \oplus (x, \nabla^g_{\dot{x}} \dot{x}) .$$

This effectively introduces curve equivalence up to covariant acceleration. In coordinates,

$$\Gamma_g : (x^i, \dot{x}^i, \ddot{x}^i) \to (x^i, \dot{x}^i) \oplus (x^i, \ddot{x}^i + \dot{x}^h \dot{x}^r \Gamma^i{}_{hr}) .$$

Evidently we can use this isomorphism over (M, g) to bring to T^2M the natural structures in $TM \oplus TM$, summarised by:

THEOREM 1. – (i) *$T^2M \to M$ is a Riemannian vector bundle with respect to $(g \times g) \circ \Gamma_g$.*

(ii) *(T^2M, T^2g) is a Riemannian $3m$-manifold with Riemannian metric*

$$T^2g = (Tg \times Tg) \circ T\Gamma_g . \quad \#$$

We need to distinguish the two aspects of induced geometry for T^2M. Firstly,

$$\Gamma_g^{-1}(\nabla^g \oplus \nabla^g)\Gamma_g = \dot{\nabla}^g : S(TM)\times S(T^2M) \to S(T^2M)$$

is the induced connection as a covariant derivation of sections of T^2M with respect to sections of TM. This derivation is compatible with $\dot{g} = (g\times g)\circ\Gamma_g$ in the sense that for all $X \in S(TM)$, $A, B \in \in S(T^2M)$ we have:

$$X\dot{g}(A, B) = \dot{g}(\dot{\nabla}^g{}_X A, B) + \dot{g}(A, \dot{\nabla}^g{}_X B)\,.$$

Secondly, T^2g induces a Levi-Civita connection ∇^{T^2g} on T^2M. This is a covariant derivation

$$\nabla^{T^2g}\colon S(T^2M)\times S(T^2M) \to S(T^2M)$$

compatible with T^2g; so, for $F, A, B \in S(T^2M)$ we have:

$$FT^2g(A, B) = T^2g(\nabla^{T^2g}{}_F A, B) + T^2g(A, \nabla^{T^2g}{}_F B)\,.$$

$\dot{\nabla}^g$ induces a rank m horizontal distribution D on TT^2M; also ∇^{T^2g} induces a rank m horizontal distribution D_1 on TT^2M, which is the orthogonal complement with respect to T^2g to the tangent space of fibres of the projection $T^2M \to M$.

If one considers the space D_2 in $T(TM \oplus TM)$ consisting of the horizontal vectors (X, Y) with respect to $\nabla^g \oplus \nabla^g$ such that $\pi_{TM}(X) = \pi_{TM}(Y)$, it follows from the constructions that D and D_1 coincide with the inverse image by $T\Gamma_g$ of D_2.

We can conveniently span this latter at

$$(x^i, \dot{x}^i) \oplus (x^i, \ddot{x}^i + \dot{x}^h\dot{x}^r\Gamma^i{}_{hr}) = (x^i, \dot{x}^i) \oplus (x^i, \dot{y}^i) \in TM \oplus TM$$

by the basis:

$$\{h_i \oplus \bar{h}_i\} = \{(\partial_i - \dot{x}^j\Gamma^k{}_{ij}\,\dot{\partial}_k) \oplus (\partial_i - \dot{y}^j\Gamma^k{}_{ij}\,\dot{\partial}_k)\}\,.$$

The complementary vertical subspace is spaned by

$$\{(\dot{\partial}_i \oplus 0),\ (0 \oplus \dot{\partial}_i)\}\,.$$

Bringing these back to TT^2M by Γ_g^{-1} we have a horizontal subspace spanned by

$$\{H_i\} = T\Gamma_g^{-1}\{h_i \oplus \tilde{h}_i\} =$$
$$= \{\partial_i - \dot{x}^j \Gamma^k{}_{ji} \dot{\partial}_k + [2\dot{x}^t \dot{x}^r \Gamma^k{}_{ti} \Gamma^t{}_{ri} - \dot{y}^j \Gamma^k{}_{ji} - \dot{x}^t \dot{x}^r \partial_i \Gamma^k{}_{tr}] \ddot{\partial}_k\}$$

and a complementary vertical frame field

$$\{\varepsilon_i, \ddot{\partial}_i\} = \{\dot{\partial}_i - 2\dot{x}^r \Gamma^k{}_{ri} \ddot{\partial}_k, \ddot{\partial}_i\}.$$

With respect to these frame fields $\{H_i, \varepsilon_i, \ddot{\partial}_i\}$ the components of $T^2 g$ appear as

$$\operatorname{diag}(2g_{ij}, g_{ij}, g_{ij}).$$

Another way to view the induced connection for T^2M is via the direct sum of two copies of the almost product structure induced by ∇^g over TM. This yields an almost product structure over T^2M and hence the connection; de Leon and Vázquez-Abal [6] give details of this viewpoint.

Bowman [1] studied the submanifold ${}^2M \subset TTM$ on which $T\pi_{TM}$ and π_{TTM} agree. This is clearly a vector bundle over TM, and in the presence of the connection ∇^g on M there is a diffeomorphism of ${}^2M$ with $TM \oplus TM$ over M. In fact Bowman's image of ${}^2M$ in $TM \oplus TM$ coincides with the image there of T^2M by our map Γ_g. Hence ${}^2M$ and T^2M are isomorphic as vector bundles over M and under the isomorphism the natural metric and connection structures correspond.

Bowman viewed the Riemannian structure on ${}^2M$ as a second order metric on M. He showed that its autoparallel curves are not necessarily geodesics of (M, g) and a corresponding result was demonstrated by the second author for T^2M. It is clear that geodesics of (M, g) are always autoparallel curves of ${}^2M$, or T^2M.

Each autoparallel curve of ${}^2M$ defines a second order exponential map and Bowman showed that it is locally a diffeomorphism and restricts to the usual exponential map; the same is now obtained for T^2M by virtue of our mutual isometries through $TM \oplus TM$.

We shall find it useful to introduce the change of coordinate matrix $L = [L_{AB}]$ induced by the change of frame isomorphism

$$\{\partial_i, \dot{\partial}_i, \ddot{\partial}_i\} \to \{H_i, \varepsilon_i, \ddot{\partial}_i\} = \{\Delta_A\}$$

The components of L^{-1} will be denoted $[L^{AB}]$. Here, of course, A, B run through $1, 2, \ldots, 3m$. The structure constants of the

Lie Algebra referred to the adapted frame $\{\Delta_A\}$ are denoted $C_{BC}{}^A$, so

$$[\Delta_B, \Delta_C] = C_{BC}{}^A \Delta_A .$$

It follows that the possible nonzero structure constants are given by

$$C_{i\bar{j}}{}^{\bar{k}} = \Gamma^k{}_{ij} , \qquad C_{ij}{}^{\bar{k}} = R_{ijh}{}^k \dot{x}^h ,$$

$$C_{ij}{}^{\bar{\bar{k}}} = (\dot{x}^h \dot{x}^r \Gamma^l{}_{hr} + \ddot{x}^l) R_{ijl}{}^k , \qquad C_{i\bar{\bar{j}}}{}^{\bar{\bar{k}}} = \Gamma^k{}_{ij} ,$$

where $\bar{k} = k + m$, $\bar{\bar{k}} = k + 2m$ and $R_{ijh}{}^k$ are the local components of the curvature whose expression is

$$R_{ijh}{}^k = \partial_j \Gamma^k{}_{ih} - \partial_i \Gamma^k{}_{jh} + \Gamma^k{}_{jl} \Gamma^l{}_{ih} - \Gamma^k{}_{il} \Gamma^l{}_{jh} .$$

We shall denote the Chrystoffel symbols of the metric connection ∇^{T^2g} with respect to the frame $\{\Delta_A\}$ by ${}^{T^2g}\Gamma^A{}_{BC}$. Of course, this connection is torsion free but with respect to this frame we have

$${}^{T^2g}\Gamma^A{}_{BC} - {}^{T^2g}\Gamma^A{}_{CB} = C_{BC}{}^A .$$

Therefore since $\nabla^{T^2g} T^2 = 0$,

$${}^{T^2g}\Gamma^A{}_{BC} = \tfrac{1}{2} \{ T^2 g^{AD} [\Delta_B (T^2 g_{DC}) + \Delta_C (T^2 g_{DB}) - \Delta_D (T^2 g_{BC})] + $$
$$+ [C_{BC}{}^A + B^A{}_{BC} + B^A{}_{CB}] \} ,$$

with

$$B^A{}_{BC} = T^2 g^{AD} \, T^2 g_{EC} \, C_{DB}{}^E .$$

In order to compute ${}^{T^2g}\Gamma^A{}_{BC}$ we need to know the $B^A{}_{BC}$; it turns out that the possible nonzero components as follows

$$B^k{}_{i\bar{j}} = \tfrac{1}{2} R^k{}_{ihj} \dot{x}^h , \qquad B^k{}_{\bar{i}\bar{j}} = \tfrac{1}{2} g^{kt} g_{rj} \Gamma^r{}_{ti} ,$$

$$B^k{}_{i\bar{\bar{j}}} = \tfrac{1}{2} (\dot{x}^h \dot{x}^r \Gamma^l{}_{hr} + \ddot{x}^l) R^k{}_{ilj} , \qquad B^k{}_{\bar{\bar{i}}\bar{\bar{j}}} = \tfrac{1}{2} g^{kt} g_{rj} \Gamma^r{}_{ti} ,$$

$$B^{\bar{k}}{}_{i\bar{j}} = - g^{kt} g_{rj} \Gamma^r{}_{ti} , \qquad B^{\bar{\bar{k}}}{}_{i\bar{\bar{j}}} = - g^{kt} g_{rj} \Gamma^r{}_{ti} ,$$

where $R^k{}_{ijh} = g^{kt} g_{hs} R_{tij}{}^s$.

The possible nonzero components ${}^{T^2g}\Gamma^{A}{}_{BC}$ are found to be:

$$ {}^{T^2g}\Gamma^{k}{}_{ij} = \Gamma^{k}{}_{ij}\,, \qquad {}^{T^2g}\Gamma^{k}{}_{i\bar{j}} = \tfrac{1}{4} R^{k}{}_{ihj}\dot{x}^{h}\,, $$

$$ {}^{T^2g}\Gamma^{k}{}_{\bar{i}j} = \tfrac{1}{4} R^{k}{}_{jhi}\dot{x}^{h}\,, \qquad {}^{T^2g}\Gamma^{k}{}_{i\bar{\bar{j}}} = \tfrac{1}{4}(\dot{x}^{h}\dot{x}^{r}\Gamma^{l}{}_{hr} + \ddot{x}^{l})R^{k}{}_{ilj}\,, $$

$$ {}^{T^2g}\Gamma^{k}{}_{\bar{\bar{i}}j} = \tfrac{1}{4}(\dot{x}^{h}\dot{x}^{r}\Gamma^{l}{}_{hr} + \ddot{x}^{l})R^{k}{}_{jli}\,, $$

$$ {}^{T^2g}\Gamma^{\bar{k}}{}_{ij} = \tfrac{1}{2} R_{ijh}{}^{k}\dot{x}^{\bar{h}}\,, \qquad {}^{T^2g}\Gamma^{\bar{k}}{}_{i\bar{j}} = \Gamma^{k}{}_{ij}\,, $$

$$ {}^{T^2g}\Gamma^{\bar{\bar{k}}}{}_{ij} = \tfrac{1}{2}(\dot{x}^{h}\dot{x}^{r}\Gamma^{l}{}_{hr} + \ddot{x}^{l})R_{ijl}{}^{k}\,, \qquad {}^{T^2g}\Gamma^{\bar{\bar{k}}}{}_{i\bar{\bar{j}}} = \Gamma^{k}{}_{ij}\,. $$

Walker [10] introduced the following terminology for a distribution $\mathfrak{D}$ on a tangent bundle in the presence of a connection:

(*a*) $\mathfrak{D}$ is *semi-parallel* if parallel relative to itself.

(*b*) $\mathfrak{D}$ is *path-parallel* if its integral curves contain all of the autoparallel curves with initial vector in $\mathfrak{D}$.

It is clear how to adapt these for other vector bundles and get an analogous result to that of Mok [7] on the frame bundle.

THEOREM 2. – *On (T^2M, T^2g) the Levi-Civita connection ∇^{T^2g} induces the splitting $TT^2M \cong \mathcal{H} \oplus \mathcal{V}$*

i) *$\mathcal{H}$ is semi-parallel if and only if (M, g) is flat.*

ii) *$\mathcal{H}$ is path-parallel.*

iii) *$\mathcal{V}$ is parallel along $\mathcal{H}$ if and only if (M, g) is flat.*

iv) *$\mathcal{V}$ is semi-parallel and path-parallel.*

v) *$\mathcal{V}$ is parallel if and only if (M, g) is flat.*

PROOF. – We use the reformulation by Yano [11] of Walker's definitions in terms of components of the connection ∇^{T^2g}. Our computation of the ${}^{T^2g}\Gamma^{A}{}_{BC}$ then yields the result. #

We consider next the case that (M, g) is flat, then it follows that (TM, Tg) is flat. Also the ∇^{T^2g}-horizonral distribution is integrable, because then $C_{BC}{}^{A} = 0$, and we can consider its integral manifolds.

THEOREM 3. – *If (M, g) is flat then $W(z_0)$ the maximal integral manifold of the ∇^{T^2g}-horizontal distribution $\mathcal{H}$ through each $z_0 \in T^2M$, is well defined, autoparallel and totally geodesic in (T^2M, T^2g).*

PROOF. – Since $\mathcal{H}$ is integrable, then any curves $c(t)$ whose tangent vectors all belong to $\mathcal{H}$ through $c(0)$. By Theorem 2, $\mathcal{H}$ is semi-parallel and hence $W(z_0)$ is autoparallel. Additionally, $\mathcal{H}$ is path-parallel and it follows that $W(z_0)$ is totally geodesic. #

2. – Second order harmonic fibrations.

THEOREM 4. – *The diffeomorphism over* (M, g)

$$\Gamma_g\colon\ (T^2M,\ T^2g) \to (TM \oplus TM,\ Tg \times Tg)$$

is harmonic.

PROOF. – This is simply because we have arranged for Γ_g to be an isometry, which is therefore harmonic by Eells and Sampson [3]. #

We now consider the more suitable $G = \frac{1}{2}T^2g$

THEOREM 5. – *The vector bundle* $\pi_2^M\colon (T^2M, G) \to (M, g)$ *is a harmonic fibration with autoparallel fibres.*

PROOF. – Evidently π_2^M is a Riemannian fibration. Then it is harmonic if and only if its fibres are minimal submanifolds, by Eells and Sampson [3]. From Yano [11] we find that the Levi-Civita connection ∇^G has semi-parallel vertical distribution $\mathcal{V}$ because

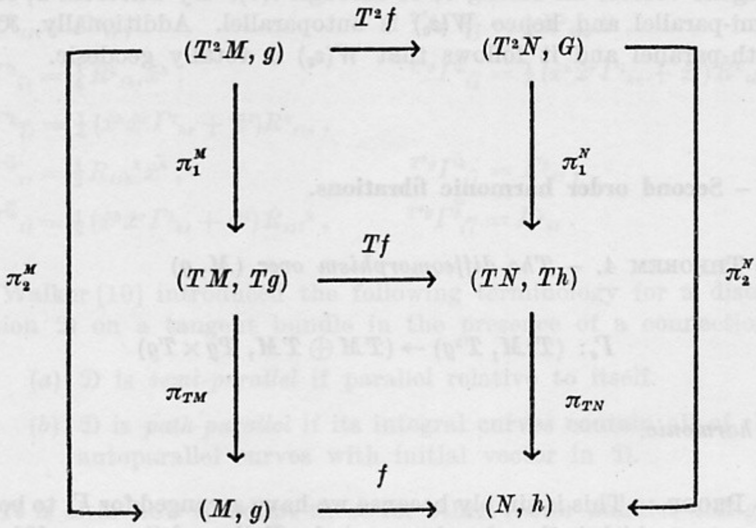
$${}^{T^2g}\Gamma^{\bar{\bar{k}}}{}_{ij} = {}^{T^2g}\Gamma^{\bar{\bar{k}}}{}_{\bar{i}\bar{j}} = {}^{T^2g}\Gamma^{k}{}_{\bar{i}\bar{j}} = {}^{T^2g}\Gamma^{\bar{\bar{k}}}{}_{i\bar{j}} = 0$$

(note: ${}^{T^2g}\Gamma^{A}{}_{BC} = {}^{G}\Gamma^{A}{}_{BC}$).

Also, $\mathcal{V}$ is integrable. Now, by Kobayashi and Nomizu [5], p. 53, the fibre of π_2^M, being integral manifolds of $\mathcal{V}$, are autoparallel. Hence the fibres are totally geodesic by [5], p. 56, therefore minimal submanifolds. #

COROLLARY. – *The vector bundle* $(T^2M, G) \to (TM, Tg)$ *is a harmonic fibration, by restriction of the fibres of* π_2^M. #

THEOREM 6. – *In this diagram*



we have:

i) $f\circ\pi_{TM}=\pi_{TN}\circ Tf$ *is harmonic if and only if* f *is harmonic.*

ii) $f\circ\pi_2^M=\pi_2^N\circ T^2f$ *is harmonic if and only if* f *is harmonic.*

iii) $f\circ\pi_2^M=\pi_2^N\circ T^2f$ *is harmonic if* (N, h) *is flat and* T^2f *is harmonic.*

PROOF. – The results i), ii) follow from Eells and Sampson [3] because π_{TM} and π_2^M are harmonic Riemannian submersions. By direct computations, π_2^N is totally geodesic if (N, h) is flat; then its left composition with harmonic T^2f yields a harmonic map. #

COROLLARY 1. – *Suppose that* (N, h) *is* $\mathbb{R}$, (M, g) *is compact and that* (T^2M, G) *is complete with nonnegative sectional curvature. Then, if* f *or* T^2f *is harmonic, it follows that the real function*

$$f\circ\pi_2^M\colon\ T^2M\to\mathbb{R}$$

is constant if it has bounded energy.

PROOF. – Harmonicity of f or T^2f is sufficient to ensure harmonicity of $f\circ\pi_2^M$ and then we have a special case of a theorem of Greene and Wu [4]. #

We note that this result may be relevant to the study of second order lagrangians, particularly for physics on S^2, S^3, S^4 for example.

COROLLARY 2. – *Suppose that (N, h) is compact with nonpositive sectional curvature, (M, g) is compact and (T^2M, G) is complete with nonnegative Ricci curvature. Then, if f is harmonic, it follows that*

$$f \circ \pi_2^M \colon T^2M \to N ,$$

is constant if it has bounded energy.

PROOF. – As for Corollary 1 but using a result from Schoen y Yau [9]. #

COROLLARY 3. – *If the horizontal distribution in T^2N is integrable, then $f \circ \pi_2^M$ is harmonic if $T^2 f$ is harmonic.*

PROOF. – This follows because π_2^N is totally geodesic if and only if the horizontal distribution in T^2N is integrable. Then its left composition with harmonic $T^2 f$ yields a harmonic map and $f \circ \pi_2^M = \pi_2^N \circ T^2 f$. #

REFERENCES

[1] R. H. BOWMAN, *Second order connections*, J. Differential Geom., **7** (1972), 549-561.

[2] C. T. J. DODSON - M. RADIVOIOVICI, *Tangent and frame bundles of order two*, An Stünt. Univ. « Al. I. Cuza » Iasi Sect. Ia Mat., **28** (1982), 62-71.

[3] J. EELLS - J. H. SAMPSON, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., **86** (1964), 109-160.

[4] R. E. GREENE - H. H. WU, *Integrals of subharmonic functions on manifolds of nonnegative curvature*, Invent. Math., **27** (1974), 265-298.

[5] S. KOBAYASHI - K. NOMIZU, *Foundations of differential geometry*, Vol. II, Interscience, New York, 1969.

[6] M. DE LEON - M. E. VAZQUEZ-ABAL, *On the geometry of the tangent bundle of order* 2, An. Univ. Bucuresti Mat., **34** (1985), 40-48.

[7] K.-P. MOK, *On the differential geometry of frame bundles of Riemannian manifolds*, J. Reine Angew. Math., **302** (1978), 16-31.

[8] S. SASAKI, *On the differential geometry of tangent bundles of Riemannian manifolds*, Tohoku Math. J., **10** (1958), 338-354.

[9] N. SCHOEN - S. T. YAU, *Harmonic maps and the topology of stable

hypersurfaces of nonnegative Ricci curvature, Comm. Math. Helvet., **51** (1976), 333-341.

[10] A. G. WALKER, *Connexions for parallel distributions in the large - II*, Quart. J. Math. Oxford, (2) **9** (1958), 221-231.

[11] K. YANO, *Differential geometry on complex and almost complex spaces*, Pergamon Press, New York, 1965.

C. T. J. Dodson: Department of Mathematics,
University of Lancaster (England)

M. E. Vázquez-Abal: Department of Geometry and Topology,
University of Santiago de Compostela (Spain)

Pervenuta in Redazione
il 17 aprile 1989