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HARMONIC GEODESIC SYMMETRIES

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We prove that a Riemannian manifold is locally symmetric if and only if all local geodesic symmetries are harmonic.

1. INTRODUCTION

Let (M,g) be a smooth n-dimensional Riemannian manifold and m a point of M. Consider an orthonormal basis, $\{e_1,\ldots,e_n\}$, of T_mM and denote by $(x^1,\ldots,x^n)$ the system of normal coordinates centered at the point m with $\frac{\partial}{\partial x^i}(m) = e_i$, $i = 1,\ldots,n$.

If ξ is a unit tangent vector at m, and γ is the geodesic $r \longmapsto \exp_m(r\xi)$ through $m = \gamma(0)$ with tangent vector $\xi = \gamma'(0)$ and arc length r, we define the map

$$\varphi_m : \exp_m(r\xi) \longrightarrow \exp_m(-r\xi) : (x^i) \longmapsto (-x^i) .$$

For each m there exists a neighborhood of m such that φ_m is a local diffeomorphism and in what follows we will always restrict to such a domain. The map φ_m is called the <u>local geodesic symmetry centered at</u> m.

This map may be used to define special classes of Riemannian manifolds. The classical example is that of a locally symmetric space. Indeed, it is well-known that a Riemannian manifold is <u>locally symmetric</u> if and only if each local geodesic symmetry is an isometry. If ∇ denotes the Riemannian connection and R the associated curvature tensor, then the above condition is equivalent to $\nabla R = 0$. A very useful criterion is given in the following

<u>LEMMA 1</u>. (M,g) <u>is locally symmetric if and only if</u>

$$\nabla_X R_{XYXY} = 0$$

<u>for all tangent vectors</u> X,Y.

For a proof of this lemma we refer to [1],[5],[7].

Next, let (M,g) and (N,h) be two Riemannian manifolds with metrics g and h and let $f : (M,g) \longrightarrow (N,h)$ be a smooth map. The pullback f^*h is a semidefinite symmetric covariant tensor of order two, called the first fundamental form. Further, the covariant differential $\nabla(df)$ is a symmetric tensor of order two with values in $f^{-1}(TN)$, called the second fundamental form of f (see [2],[3]). A map with vanishing second fundamental form is said to be totally geodesic.

The trace of $\nabla(df)$ is denoted by $\tau(f)$ and is called the tension field of f. A map with vanishing tension field is called a harmonic map.

If $\mathcal{U} \subset M$ is a domain with coordinates $(x^1,\ldots,x^m)$ and $\mathcal{V} \subset N$ is a domain with coordinates $(y^1,\ldots,y^n)$ such that $f(\mathcal{U}) \subset \mathcal{V}$, then f can be locally represented by $y^\alpha = f^\alpha(x^1,\ldots,x^m)$, $\alpha = 1,\ldots,n$. The metric tensor g is represented by $g(x) = g_{ij}(x)dx^i dx^j$, $i,j = 1,\ldots,m$, and similarly we have $h(y) = h_{\alpha\beta}(y)dy^\alpha dy^\beta$, $\alpha,\beta = 1,\ldots,n$. $df(x)$ is represented by the matrix $(\frac{\partial f^\alpha}{\partial x^i})$. In this case we have

$$(f^*h)_{ij} = \frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j} h_{\alpha\beta} ,$$

$$(\nabla(df))^\gamma_{ij} = \frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - {}^M\Gamma^k_{ij}\frac{\partial f^\gamma}{\partial x^k} + {}^N\Gamma^\gamma_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j} ,$$

where ${}^M\Gamma^k_{ij}$ and ${}^N\Gamma^k_{ij}$ are the Christoffel symbols for (M,g) and (N,h) respectively.

It follows that f is harmonic if and only if

$$g^{ij}(\nabla(df))^\gamma_{ij} = 0 . \tag{1}$$

Finally, we get easily from (1)

LEMMA 2. For the geodesic symmetry φ_m we have

$$\nabla(d\varphi_m)^k_{ij}(p) = {}^M\Gamma^k_{ij}(p) + {}^N\Gamma^k_{ij}(-p) , \tag{2}$$

$$\tau(\varphi_m)^k(p) = g^{ij}(p)\{{}^M\Gamma^k_{ij}(p) + {}^N\Gamma^k_{ij}(-p)\} . \tag{3}$$

2. HARMONIC GEODESIC SYMMETRIES

To prove our main result we need the expressions for g_{ij} and g^{ij} with respect to normal coordinates $(x^1,\ldots,x^n)$. Let $p = \exp_m(r\xi)$ where ξ is a unit vector. Then we have (see [1],[4],[6]) :

$$g_{ij}(p) = \delta_{ij} - \frac{r^2}{3} R_{\xi i\xi j}(m) - \frac{r^3}{6}\nabla_\xi R_{\xi i\xi j}(m) + O(r^4) , \tag{4}$$

$$g^{ij}(p) = \delta_{ij} + \frac{r^2}{3} R_{\xi i\xi j}(m) + \frac{r^3}{6}\nabla_\xi R_{\xi i\xi j}(m) + O(r^4) . \tag{5}$$

Also we note that

$$ {}^M\Gamma^k_{ij} = \frac{1}{2} g^{k\ell}\left(\frac{\partial g_{i\ell}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell}\right) . \tag{6}$$

Now we are ready to prove

THEOREM. (M,g) is locally symmetric if and only if each geodesic symmetry φ_m, $m \in M$, is harmonic.

Proof. First, suppose that (M,g) is locally symmetric. Then each φ_m is isometric and hence harmonic.

To prove the converse we write (4) as follows :

$$g_{ij}(p) = \delta_{ij} - \frac{1}{3}x^k x^\ell R_{ki\ell j}(m) - \frac{1}{6}x^k x^\ell x^s \nabla_k R_{\ell isj}(m) + O(r^4) .$$

Then a straightforward computation, using (6), shows that

$$\begin{aligned} {}^M\Gamma^k_{ij}(p) = &-\frac{r}{3}(R_{ik\xi j} + R_{jk\xi i})(m) \\ &- \frac{r^2}{12}(\nabla_i R_{\xi k\xi j} + \nabla_j R_{\xi i\xi k} + 2\nabla_\xi R_{ik\xi j} + 2\nabla_\xi R_{\xi ijk} - \nabla_k R_{\xi i\xi j})(m) \\ &+ O(r^3) . \end{aligned} \tag{7}$$

Using the Bianchi identities we get from (7)

$$(8)\quad {}^{M}\Gamma^{k}_{ij}(p) + {}^{M}\Gamma^{k}_{ij}(-p) = -\frac{r^2}{6}\,(3\nabla_{\xi}R_{ik\xi j} + 3\nabla_{\xi}R_{\xi ijk} + \nabla_{k}R_{\xi j\xi i})(m) + O(r^4)\ .$$

Moreover, it is easily seen that the left hand side is an even function of r. Hence we may put

$${}^{M}\Gamma^{k}_{ij}(p) + {}^{M}\Gamma^{k}_{ij}(-p) = r^2\alpha_{2ij} + r^4\alpha_{4ij} + O(r^6)\ .$$

Finally, using (1), (3) and (5) we get the following condition for a harmonic geodesic symmetry φ_m :

$$\sum_{i,j=1}^{n} \left(\delta_{ij} + \frac{r^2}{3}R_{\xi i\xi j}(m) + \frac{r^3}{6}\nabla_{\xi}R_{\xi i\xi j}(m) + O(r^4)\right)\left(r^2\alpha_{2ij} + r^4\alpha_{4ij} + O(r^6)\right) = 0\ .$$

Hence we have the following necessary conditions :

$$(9)\quad \left\{\begin{array}{l} \sum_{i}\alpha_{2ii} = 0\ , \\ \sum_{i,j} R_{\xi i\xi j}(m)\alpha_{2ij} + 3\sum_{i}\alpha_{4ii} = 0\ , \\ \sum_{i,j}\nabla_{\xi}R_{\xi i\xi j}(m)\alpha_{2ij} = 0\ . \end{array}\right.$$

Using (8), the third condition in (9) becomes

$$(10)\quad \sum_{i,j}\nabla_{\xi}R_{\xi i\xi j}(m)(3\nabla_{\xi}R_{ik\xi j} + 3\nabla_{\xi}R_{\xi ijk} + \nabla_{k}R_{\xi j\xi i})(m) = 0\ .$$

Since e_k is arbitrary we may put $e_k = \xi$. Then, from (10) we obtain

$$\sum_{i,j}(\nabla_{\xi}R_{\xi i\xi j})^2(m) = 0$$

and hence

$$\nabla_{\xi}R_{\xi i\xi j} = 0\ .$$

Now the result follows at once from Lemma 1.

As a special case we have

COROLLARY. (M,g) is locally symmetric if and only if each local geodesic symmetry is a totally geodesic map.

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