Gaussian Manifold and Geodesics

Normal Distributions: N(μ,σ)

The Normal probability density function is given by $N(\mu, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$,

where μ is the mean, and σ is the standard deviation. The Fisher metric for the Normal Manifold NM with respect to the coordinate system (μ, σ) is given by

$g = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{pmatrix}$, and the Christoffel symboles are given by :

$$\Gamma_{11}{}^{1} = 0;$$
$$\Gamma_{12}{}^{1} = \Gamma_{21}{}^{1} = \frac{-1}{\sigma};$$
$$\Gamma_{22}{}^{1} = 0;$$
$$\Gamma_{11}{}^{2} = \frac{1}{2\sigma};$$
$$\Gamma_{12}{}^{2} = \Gamma_{21}{}^{2} = 0;$$
$$\Gamma_{22}{}^{2} = \frac{-1}{\sigma}.$$

Finding Geodesics Numerically for the Normal Manifold:

A geodesic on *a* manifold *M* is *a* curve $C(t) = (x(t),\ y(t))$ such that :

$$x'' + \Gamma_{11}{}^{1}(x')^2 + 2\,\Gamma_{12}{}^{1}\,x'\,y' + \Gamma_{22}{}^{1}(y')^2 = 0 \text{ and}$$

$$y'' + \Gamma_{11}{}^{2}(x')^2 + 2\,\Gamma_{12}{}^{2}\,x'\,y' + \Gamma_{22}{}^{2}(y')^2 = 0;$$ where the $\Gamma_{jk}{}^{i}$ are Christoffel symboles.

On the Normal Manifold the geodesic equations are reperesented by :

$$x'' = \frac{2}{\sigma}\,x'\,y'; \text{ and}$$

$$y'' = \frac{-1}{2\sigma}(x')^2 + \frac{1}{\sigma}(y')^2.$$ They are independent of the mean μ.

… ..

$$x''[t] == \frac{2}{\sigma} x'[t] * y'[t]$$

$$y''[t] == \frac{-1}{2\sigma} (x'[t])^2 + \frac{1}{\sigma} (y'[t])^2$$

$$x''[t] == \frac{2 x'[t] y'[t]}{\sigma}$$

$$y''[t] == -\frac{x'[t]^2}{2\sigma} + \frac{y'[t]^2}{\sigma}$$

Solving the geodesics equations for Normal manifold:???

$$x''(t) == \frac{2 x'(t) y'(t)}{\sigma}$$

$$y''(t) == \frac{y'(t)^2}{\sigma} - \frac{x'(t)^2}{2\sigma}$$

$$x''[t] == \frac{2 x'[t] y'[t]}{\sigma}$$

$$y''[t] == -\frac{x'[t]^2}{2\sigma} + \frac{y'[t]^2}{\sigma}$$

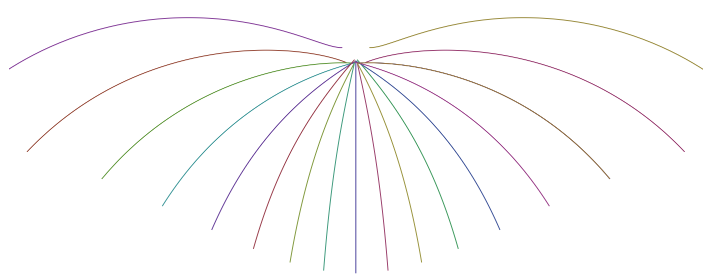
$$\text{DSolve}\left[\left\{\sigma\, x''(t) == 2 x'(t) y'(t),\ \sigma\, y''(t) == y'(t)^2 - \frac{1}{2} x'(t)^2\right\}, \{x(t), y(t)\}, t\right]$$

$$\text{DSolve}\left[\left\{\sigma x''[t] == 2 x'[t] y'[t],\ \sigma y''[t] == -\frac{1}{2} x'[t]^2 + y'[t]^2\right\}, \{x[t], y[t]\}, t\right]$$

1)Normal geodesics (x(t),y(t)) where σ**=1** centered at the origin:(**N(**μ**,1)**)

In[66]:=
```
With[{μ = μ, σ = 1},
  normal = Flatten[Table[NDSolve[{x''[t] == 2/σ x'[t] * y'[t], y''[t] ==
        -1/(2 σ) (x'[t])^2 + 1/σ (y'[t])^2, x[0] == 0, y[0] == 0, x'[0] == Cos[a],
        y'[0] == Sin[a]}, {x, y}, {t, 1, 10}], {a, 0, 2 Pi, Pi / 12}], 1]];
NG[μ , 1] = ParametricPlot[Evaluate[{x[t], y[t]} /. normal],
  {t, 0, 2.4}, AspectRatio → Automatic, Axes → False,
  PlotRange → {{-2, 2}, {-1.5, .3}}, Prolog → AbsoluteThickness[1]]
```

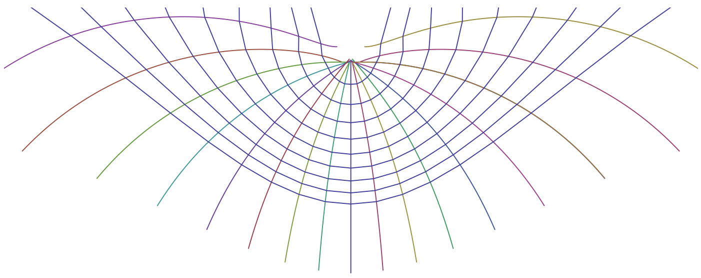
Out[67]=



- Circles for the normal geodesics NG[μ,1]:

In[61]:=
```
With[{μ = μ, σ = 1},
  normal = Flatten[Table[NDSolve[{x''[t] == 2/σ x'[t] * y'[t], y''[t] ==
        -1/(2 σ) (x'[t])^2 + 1/σ (y'[t])^2, x[0] == 0, y[0] == 0, x'[0] == Cos[a],
        y'[0] == Sin[a]}, {x, y}, {t, 1, 10}], {a, 0, 2 Pi, Pi / 12}], 1]];
NG[μ , 1] = ParametricPlot[Evaluate[{x[t], y[t]} /. normal],
   {t, 0, 2.4}, AspectRatio → Automatic, Axes → False,
   PlotRange → {{-2, 2}, {-1.5, .3}}, Prolog → AbsoluteThickness[1]];
Do[vc[i] = ListPlot[Evaluate[{x[i / 7], y[i / 7]} /. normal],
    Joined → True, AspectRatio → Automatic, Axes → False,
    PlotRange → {{-2, 2}, {-1.5, .5}}, PlotStyle → {SpecCol[i]},
    (*include this line for color*)Prolog → AbsoluteThickness[1],
    DisplayFunction → Identity], {i, 14}];
B = Show[vc[1], vc[2], vc[3], vc[4], vc[5], vc[6], vc[7], vc[8],
   vc[9], DisplayFunction → $DisplayFunction, Axes → Automatic];
Show[NG[μ , 1], B]
```

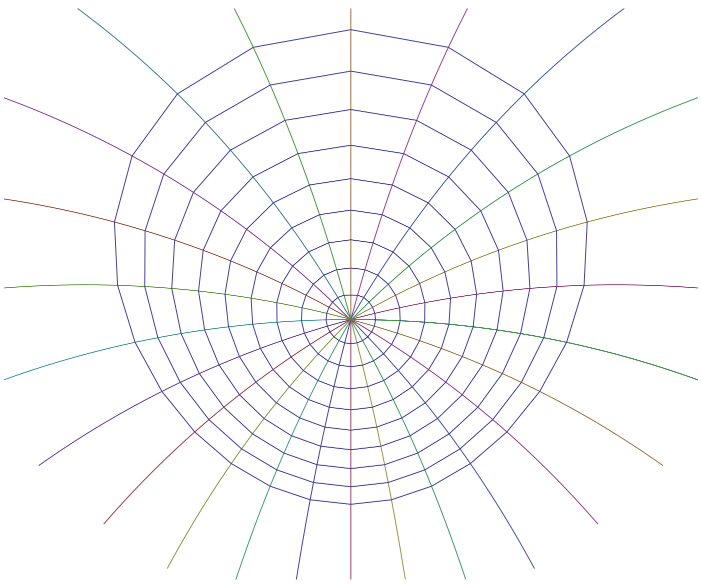
Out[65]=



2)Normal geodesics (x(t),y(t)) where σ**=3** centered at the origin:(**N(**μ**,3)**)

In[68]:=
```
With[{μ = μ, σ = 3},
  normal = Flatten[Table[NDSolve[{x''[t] == 2/σ x'[t] * y'[t], y''[t] ==
        -1/(2 σ) (x'[t])^2 + 1/σ (y'[t])^2, x[0] == 0, y[0] == 0, x'[0] == Cos[a],
        y'[0] == Sin[a]}, {x, y}, {t, 1, 10}], {a, 0, 2 Pi, Pi / 12}], 1]];
 NG[μ , 3] = ParametricPlot[Evaluate[{x[t], y[t]} /. normal],
   {t, 0, 2.4}, AspectRatio → Automatic, Axes → False,
   PlotRange → {{-2, 2}, {-1.5, 1.8}}, Prolog → AbsoluteThickness[1]];
 Do[vc[i] = ListPlot[Evaluate[{x[i / 7], y[i / 7]} /. normal],
    Joined → True, AspectRatio → Automatic, Axes → False,
    PlotRange → {{-2, 2}, {-1.5, 1.8}}, PlotStyle → {SpecCol[i]},
    (*include this line for color*)Prolog → AbsoluteThickness[1],
    DisplayFunction → Identity], {i, 14}];
 B = Show[vc[1], vc[2], vc[3], vc[4], vc[5], vc[6], vc[7], vc[8],
   vc[9], DisplayFunction → $DisplayFunction, Axes → Automatic];
 Show[NG[μ , 3], B]
```

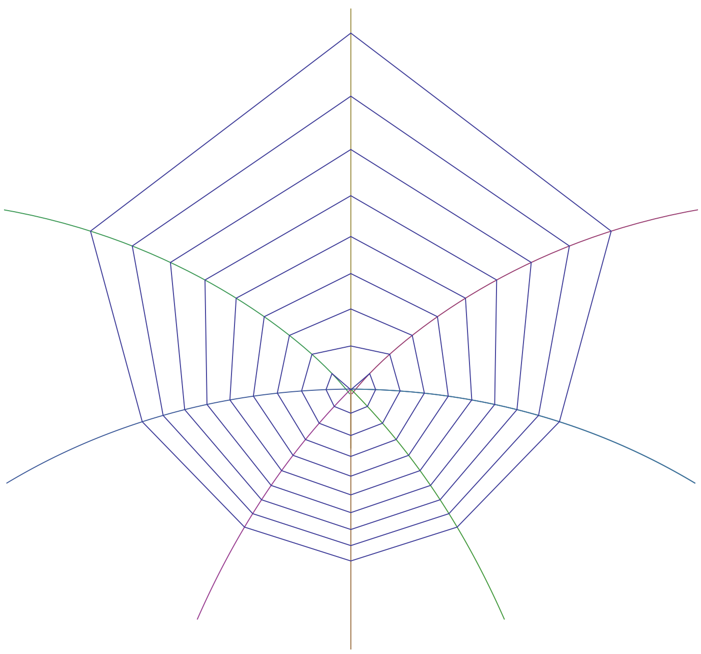
Out[72]=



3)Normal geodesics (x(t),y(t)) where σ**=2** centered at the origin:(**N(**μ**,2)**)

In[73]:=
```
With[{μ = μ, σ = 2},
  normal = Flatten[Table[NDSolve[{x''[t] == 2/σ x'[t] * y'[t], y''[t] ==
        -1/(2 σ) (x'[t])^2 + 1/σ (y'[t])^2, x[0] == 0, y[0] == 0, x'[0] == Cos[a],
        y'[0] == Sin[a]}, {x, y}, {t, 1, 10}], {a, 0, 2 Pi, Pi / 4}], 1]];
NG[μ , 2] = ParametricPlot[Evaluate[{x[t], y[t]} /. normal],
   {t, 0, 2.4}, AspectRatio → Automatic, Axes → False,
   PlotRange → {{-2, 2}, {-1.5, 2.2}}, Prolog → AbsoluteThickness[1]];
Do[vc[i] = ListPlot[Evaluate[{x[i / 7], y[i / 7]} /. normal],
    Joined → True, AspectRatio → Automatic, Axes → False,
    PlotRange → {{-2, 2}, {-1.5, 2.2}}, PlotStyle → {SpecCol[i]},
    (*include this line for color*)Prolog → AbsoluteThickness[1],
    DisplayFunction → Identity], {i, 14}];
B = Show[vc[1], vc[2], vc[3], vc[4], vc[5], vc[6], vc[7], vc[8],
   vc[9], DisplayFunction → $DisplayFunction, Axes → Automatic];
Show[NG[μ , 2], B]
```

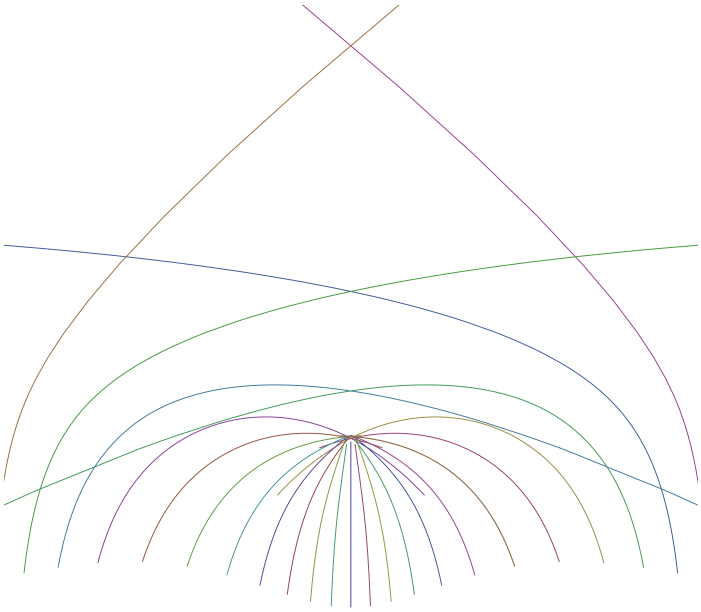
Out[77]=



4)Normal geodesics (x(t),y(t)) where σ=0.5 centered at the origin:(N(μ,.5))

In[78]:=
```
With[{μ = μ, σ = .5},
  normal = Flatten[Table[NDSolve[{x''[t] == 2/σ x'[t] * y'[t], y''[t] ==
        -1/(2 σ) (x'[t])^2 + 1/σ (y'[t])^2, x[0] == 0, y[0] == 0, x'[0] == Cos[a],
       y'[0] == Sin[a]}, {x, y}, {t, 1, 10}], {a, 0, 2 Pi, Pi / 12}], 1]];
NG[μ , .5] = ParametricPlot[Evaluate[{x[t], y[t]} /. normal],
  {t, 0, 3}, AspectRatio → Automatic, Axes → False,
  PlotRange → {{-2, 2}, {-1.3, 2.5}}, Prolog → AbsoluteThickness[1]]
```

Out[79]=



In[80]:=
```
With[{μ = μ, σ = .5},
  normal = Flatten[Table[NDSolve[{x''[t] == 2/σ x'[t] * y'[t], y''[t] ==
        -1/(2 σ) (x'[t])^2 + 1/σ (y'[t])^2, x[0] == 0, y[0] == 0, x'[0] == Cos[a],
        y'[0] == Sin[a]}, {x, y}, {t, 1, 10}], {a, 0, 2 Pi, Pi / 12}], 1]];
NG[μ , .5] = ParametricPlot[Evaluate[{x[t], y[t]} /. normal],
   {t, 0, 3}, AspectRatio → Automatic, Axes → False,
   PlotRange → {{-2, 2}, {-1.3, 2.5}}, Prolog → AbsoluteThickness[1]];
Do[vc[i] = ListPlot[Evaluate[{x[i / 7], y[i / 7]} /. normal],
    Joined → True, AspectRatio → Automatic, Axes → False,
    PlotRange → {{-2, 2}, {-1.3, 2.5}}, PlotStyle → {SpecCol[i]},
    (*include this line for color*)Prolog → AbsoluteThickness[1],
    DisplayFunction → Identity], {i, 14}];
B = Show[vc[1], vc[2], vc[3], vc[4], vc[5], vc[6], vc[7], vc[8],
   vc[9], DisplayFunction → $DisplayFunction, Axes → Automatic];
Show[NG[μ , .5], B]
```

Out[84]=