5 Product Spaces

Recall $X \times Y = \{(x,y) : x \in X, y \in Y\}$. A *rectangle* is any set $E \times F \subseteq X \times Y$.

Theorem 5.1

If $\mathcal{C}_i, i = 1,2$ are semi-rings in $X_i, i = 1,2$ respectively then $\mathcal{C}_1 \times \mathcal{C}_2$ is a semi-ring in $X_1 \times X_2$.

Proof

Let $E_1 \times F_1, E_2 \times F_2 \in \mathcal{C}_1 \times \mathcal{C}_2$. Then

$$\begin{aligned} (E_1 \times F_1) \cap (E_2 \times F_2) &= (E_1 \cap E_2) \times (F_1 \cap F_2) \\ &\in \mathcal{C}_1 \times \mathcal{C}_2 \quad \text{since } \mathcal{C}_1 \text{ and } \mathcal{C}_2 \text{ are semi-rings.} \end{aligned}$$

The result obviously extends to infinite intersections.

For differences observe that

$$\begin{aligned} (E_1 \times F_1) \setminus (E_2 \times F_2) &= ((E_1 \setminus E_2) \times (F_1 \setminus F_2)) \\ &\quad \cup ((E_1 \cap E_2) \times (F_1 \setminus F_2)) \\ &\quad \cup ((E_1 \setminus E_2) \times (F_1 \cap F_2)), \end{aligned}$$

a disjoint union. (This is most easily seen in a diagram.) And since $\mathcal{C}_1$ is a semi-ring we can write $E_1 \setminus E_2 = \bigcup_{i=3}^m E_i$ for some disjoint $E_i \in \mathcal{C}_1$. Similarly $F_1 \setminus F_2 = \bigcup_{j=3}^n F_j$ for some disjoint $F_i \in \mathcal{C}_2$. Thus

$$\begin{aligned} (E_1 \times F_1) \setminus (E_2 \times F_2) &= \bigcup_{i=3}^{m} \bigcup_{j=3}^{n} (E_i \times F_j) \\ &\quad \cup \bigcup_{j=3}^{n} ((E_1 \cap E_2) \times F_j) \cup \bigcup_{i=3}^{m} (E_i \times (F_1 \cap F_2)), \end{aligned}$$

a finite union of disjoint sets of $\mathcal{C}_1 \times \mathcal{C}_2$ as required. ■

Note If $\mathcal{E}_i, i = 1,2$ are σ-fields in X_i respectively then $\mathcal{E}_1 \times \mathcal{E}_2$ need not be a σ-field in $X_1 \times X_2$.

Definition The *product σ-field of $\mathcal{E}_1$ and $\mathcal{E}_2$*, denoted by $\mathcal{E}_1 * \mathcal{E}_2$ is the minimal σ-field containing $\mathcal{E}_1 \times \mathcal{E}_2$.

Definition For $A \subseteq X \times Y$ the *section of A at $x \in X$* is

$$A_x = \{y : (x,y) \in A\},$$

while the *section of A at $y \in Y$* is

$$A^y = \{x : (x,y) \in A\}.$$

Theorem 5.2 *Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces. Then if $A \in \mathcal{F} * \mathcal{G}$ we have*

$$A_x \in \mathcal{G} \quad \textit{for all } x \in X,$$

$$A^y \in \mathcal{F} \quad \textit{for all } y \in Y.$$

Proof Let

$$\mathcal{C} = \{E \subseteq X \times Y : E_x \in \mathcal{G} \text{ for all } x \in X\}.$$

If $G \times H \in \mathcal{F} \times \mathcal{G}$ then

$$(G \times H)_x = \begin{cases} H & \text{if } x \in \mathcal{G} \\ \emptyset & \text{if } x \notin \mathcal{G} \end{cases}$$

In both cases the result is in $\mathcal{G}$ as is required for inclusion in $\mathcal{C}$, hence $\mathcal{F} \times \mathcal{G} \subseteq \mathcal{C}$.

Claim $\mathcal{C}$ is a σ-field.

Simply note that $\left(\bigcup_{n=1}^{\infty} E_n\right)_x = \bigcup_{n=1}^{\infty} (E_n)_x$ and

$$(E_1 \setminus E_2)_x = \begin{cases} (E_1)_x & \text{if } x \in (E_1)_y \setminus (E_2)_y \\ (E_1)_x \setminus (E_2)_x & \text{if } x \in (E_1)_y \cap (E_2)_y \\ \phi & \text{otherwise.} \end{cases}$$

So, since $\mathcal{G}$ is a σ-field, we obtain the claim.

Thus $\mathcal{C}$ is **a** σ-field containing $\mathcal{F} \times \mathcal{G}$ whilst $\mathcal{F} * \mathcal{G}$ is **the** minimal such σ-field. Hence $\mathcal{F} * \mathcal{G} \subseteq \mathcal{C}$.

So if $A \in \mathcal{F} * \mathcal{G}$ then A satisfies the condition defining the collection $\mathcal{C}$, namely $A_x \in \mathcal{G}$ for all $x \in X$.

Similarly, for A^y examine $\mathcal{D} = \{E \subseteq X \times Y : E^y \in \mathcal{F} \text{ for all } y \in Y\}$. ■

(Note how the form of this proof is very similar to that of Corollary 1.5 and Theorem 1.7 in the notes.)

Our aim now is, given measure spaces $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ to define a measure on the *Product Measurable Space* $(X \times Y, \mathcal{F} * \mathcal{G})$. We shall show how to use integration to give a measure (*The Product Measure*) on this space.

Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{G}, \nu)$ be σ-finite measure spaces.

Lemma 5.1

For all $A \in \mathcal{F} * \mathcal{G}$ the ν-measure of an x-section, $\nu(A_x) : X \to \mathbb{R}^*$, is an $\mathcal{F}$-measurable function.

Proof Not given. ■

Note The situation is symmetric so $\mu(A^y) : Y \to \mathbb{R}^*$ is an $\mathcal{G}$-measurable function.

Theorem 5.3 *The set function*

$$\lambda(A) = \int_X \nu(A_x) d\mu \tag{1}$$

is a measure on $\mathcal{F} * \mathcal{G}$.

Proof Not given. ■

Notation I will write $\lambda = \nu * \mu$, though this is non-standard. But now we have a measure space $(X \times Y, \mathcal{F} * \mathcal{G}, \nu * \mu)$.

For $C \times D \in \mathcal{F} * \mathcal{G}$ we have

$$\nu((C \times D)_x) = \begin{cases} \nu(D) & \text{if } x \in C \\ 0 & \text{otherwise.} \end{cases}$$

This is a simple function so the integral (1) simply evaluates as $\lambda(C \times D) = \nu(D)\mu(C)$. So λ extends the measure $\nu \times \mu$. From Theorem 2.12, if μ and ν are σ-finite then such extensions are unique. But by symmetry, $\int_Y \mu(A^y) d\nu$ is also a measure on $\mathcal{F} * \mathcal{G}$ extending $\nu \times \mu$. So by uniqueness,

$$\int_X \nu(A_x) d\mu = \int_Y \mu(A^y) d\nu. \tag{2}$$

If $g : X \times Y \to \mathbb{R}^*$ let $g_x : Y \to \mathbb{R}^*$ be given by $g_x(y) = g(x, y)$ and $g^y : X \to \mathbb{R}^*$ by $g^y(x) = g(x, y)$. Then

Lemma 5.2

If $g : X \times Y \to \mathbb{R}^*$ *is* $\mathcal{F} * \mathcal{G}$*-measurable then* g_x *is* $\mathcal{G}$*-measurable and* g^y *is* $\mathcal{F}$*-measurable.*

Proof From the definition, g being $\mathcal{F} * \mathcal{G}$-measurable means that

$$\{(x, y) : g(x, y) > c\} \in \mathcal{F} * \mathcal{G} \quad \text{for all } c \in \mathbb{R},$$

in which case, by Theorem 5.2,

$$\{(x, y) : g(x, y) > c\}_x \in \mathcal{G} \quad \text{for all } c \in \mathbb{R},$$

and so

$$\{y : g_x(y) > c\} \in \mathcal{G} \quad \text{for all } c \in \mathbb{R}.$$

Hence g_x is $\mathcal{G}$-measurable. Similarly for g^y. ■

We now come to an important result that expresses integration with respect to a product measure in terms of iterated integrals with respect to the two original measures. It is, in fact, most often used to justify the interchange of integrals.

Theorem 5.4 (Fubini) *Let* $(X, \mathcal{F}, \mu)$ *and* $(Y, \mathcal{G}, \nu)$ *be* σ*-finite measure spaces and* $\lambda = \mu * \nu$. *Let* $g : X \times Y \to \mathbb{R}^*$ *be* $\mathcal{F} * \mathcal{G}$*-measurable.*

(i) *If g is non-negative then the functions*

$$\alpha(x) = \int_Y g_x d\nu \quad \text{and} \quad \beta(y) = \int_X g^y d\mu$$

are measurable and

$$\int_{X\times Y} g d\lambda = \int_X \left(\int_Y g_x d\nu \right) d\mu = \int_Y \left(\int_X g^y d\mu \right) d\nu. \tag{3}$$

(ii) *If $g : X \times Y \to \mathbb{R}^*$ is λ-integrable then g_x is ν-integrable for almost all x, g^y is μ-integrable for almost all y and* (3) *holds.*

(iii) *If $g : X \times Y \to \mathbb{R}^*$ is $\mathcal{F} * \mathcal{G}$-measurable and*

$$\int_X \left(\int_Y |g_x| \, d\nu \right) d\mu < \infty$$

then $g : X \times Y \to \mathbb{R}^*$ is λ-integrable.

Proof

(i) This is done in the same stages as found in the proof of Lemma 2.13. Consider first $g = \chi_A$ for some $A \in \mathcal{F} \times \mathcal{G}$. Then

$$\begin{aligned} \alpha(x) &= \int_Y (\chi_A)_x \, d\nu \\ &= \nu \{y : (\chi_A)_x (y) = 1\} \qquad \text{since } (\chi_A)_x \text{ is a simple function,} \\ &= \nu\{y : (x, y) \in A\} \\ &= \nu(A_x), \end{aligned}$$

which is measurable by Lemma 5.1. Similarly for $\beta(y)$.

We can now compare the integrals. For g we have

$$\int_{X\times Y} g d\lambda = \int_{X\times Y} \chi_A d\lambda = \lambda(A)$$

by definition of integration of a simple function such as χ_A. Also

$$\begin{aligned} \int_X \alpha(x) d\mu &= \int_X \nu(A_x) d\mu \\ &= \lambda(A) \text{ by definition (1) of } \lambda. \end{aligned}$$

Thus we get one of the equalities in (3). The other follows from using (2).

Secondly, for

$$g = \sum_{i=1}^{n} a_i \chi_{A_i},$$

a simple function, then

$$\alpha(x) = \sum_{i=1}^{n} a_i \nu(A_{ix}),$$

a finite sum of measurable functions hence measurable. Further

$$\int_{X \times Y} g d\lambda = \sum_{i=1}^{n} a_i \lambda(A_i)$$

while

$$\begin{aligned} \int_X \alpha(x) d\mu &= \sum_{i=1}^{n} a_i \int_X \nu(A_{ix}) d\mu \\ &= \sum_{i=1}^{n} a_i \lambda(A_i). \end{aligned}$$

So (3) holds for simple functions.

Finally, given a non-negative g choose a sequence of simple, measurable functions $\{g_n\}_{n\geq 1}$ increasing to g. Then $\{g_{nx}\}_{n\geq 1}$ and $\{g_n^y\}_{n\geq 1}$ are similar sequences converging to g_x and g^y respectively. We can apply Lebesgue's Monotone Convergence Theorem, obtaining

$$\alpha(x) = \int_Y g_x d\nu = \lim_{n\to\infty} \int_Y g_{nx} d\nu,$$

which is the limit of measurable functions, by the second part above, hence measurable. Similarly for $\beta(y)$.

So now $\left\{\int_Y g_{nx} d\nu\right\}_{n\geq 1}$ is an increasing sequence of non-negative measurable functions and we can apply Theorem 4.11 again. Thus

$$\begin{aligned} \int_X \alpha(x) d\mu &= \int_X \left(\lim_{n\to\infty} \int_Y g_{nx} d\nu\right) d\mu \\ &= \lim_{n\to\infty} \int_X \left(\int_Y g_{nx} d\nu\right) d\mu && \text{by Theorem 4.11,} \\ &= \lim_{n\to\infty} \int_{X\times Y} g_n d\lambda && \text{since (3) holds for simple functions,} \\ &= \int_{X\times Y} g d\lambda && \text{by Theorem 4.11 again.} \end{aligned}$$

Thus we get one of the equalities in (3). The other follows from using (2). Hence (3) holds for non-negative g.

(ii) Assuming now that g is λ-integrable implies that both g^+ and g^- are λ-integrable and in particular, $\mathcal{F} * \mathcal{G}$-measurable. Apply (i) to g^+ and g^-. Let

$$\alpha^{\pm}(x) = \int\limits_Y g_x^{\pm} d\nu. \tag{4}$$

Then (3) for non-negative functions implies

$$\begin{aligned} \int_X \alpha^{\pm}(x) d\mu &= \int_{X\times Y} g^{\pm} d\lambda \\ &< \infty \qquad \text{since } g \text{ is } \lambda\text{-integrable.} \end{aligned}$$

So, by Lemma 4.5 both α^+ and α^- are finite except, possibly, on (perhaps different) sets of μ-measure zero.

But $\alpha^{\pm}(x) < \infty$ a.e. (μ) implies

$$\int_Y g_x^{\pm} d\nu = \alpha^{\pm}(x) < \infty$$

a.e.(μ), in which case $g_x^{\pm}$ are ν-integrable a.e.(μ). So outside the union of the two sets of μ-measure zero $g_x = g_x^+ - g_x^-$ is ν-integrable. Similarly for g^y.

Now apply (3) for non-negative functions to both g^+ and g^- separately and subtract to get (3) for g.

(iii) Recall from an earlier note that if g is $\mathcal{F} * \mathcal{G}$-measurable then $|g|$ is also $\mathcal{F} * \mathcal{G}$-measurable and, trivially, it is non-negative. So by (i)

$$\begin{aligned} \int_{X\times Y} |g| d\lambda &= \int_X \left(\int_Y |g| d\nu \right) d\mu \\ &< \infty \qquad \text{by assumption.} \end{aligned}$$

So $|g|$ is λ-integrable and thus g is λ-integrable. Thus we are back to case (ii). ■

Example Let

$$g(x,y) = \begin{cases} e^{-y}\sin 2xy & \text{on } [0,1 \times [0,\infty) \\ 0 & \text{elsewhere} \end{cases}$$

Let λ be the product measure on $\mathcal{L} * \mathcal{L}$.

Claim g is λ-integrable.

Note that $|g| \le e^{-y}$ so, by Corollary 4.18 it suffices to show that $e^{-y} \in \mathcal{L}(\lambda)$. But e^{-y} is the limit of an increasing sequence of non-negative λ-measurable simple functions, for example,

$$h_N(x,y) = \sum_{n \le N^2} e^{-n/N} \chi_{A_{n,N}}$$

where

$$A_{n,N} = [0,1] \times \left[\frac{n-1}{N}, \frac{n}{N}\right].$$

Then $\lambda(A_{n,N}) = \frac{1}{N}$, that is, the set is λ-measurable. Hence e^{-y} is λ-measurable. All functions are non-negative so, by Lebesgue's Monotone Convergence Theorem,

$$\begin{aligned}
\int_{[0,1]\times[0,\infty)} e^{-y} d\lambda &= \lim_{N\to\infty} \int_{[0,1]\times[0,\infty)} h_N d\lambda \\
&= \lim_{N\to\infty} \sum_{n \le N^2} \int_{(n-1)/N}^{n/N} e^{-n/N} dy \\
&\le \lim_{N\to\infty} \sum_{n \le N^2} \int_{(n-1)/N}^{n/N} e^{-y} dy \\
&= \lim_{N\to\infty} \int_0^N e^{-y} dy \\
&= 1.
\end{aligned}$$

Hence $e^{-y} \in \mathcal{L}(\lambda)$ as required and the claim is verified.

Then by Theorem 5.4(ii) we have

$$\int_0^1 \int_0^\infty e^{-y} \sin 2xy dy dx = \int_0^\infty \int_0^1 e^{-y} \sin 2xy dy dx. \tag{5}$$

But, on integrating by parts,

$$\int_0^\infty e^{-y} \sin 2xy dy = \frac{2x}{1+4x^2}$$

so the left hand side of (5) equals

$$\int_0^1 \frac{2x}{1+4x^2} dx = \frac{1}{4} \log 5.$$

The right hand side of (5) contains

$$\int_0^1 e^{-y} \sin 2xy dy = \frac{e^{-y} \sin^2 y}{y}.$$

Hence (5) gives

$$\int_0^\infty \frac{e^{-y}\sin^2 y}{y}dy = \frac{1}{4}\log 5.$$

Note that a lot of the above example was directed at showing the function to be μ_2-integrable. This can be weakened to $\mathcal{L}^2$-measurable. It is possible to extend Fubini's result, proving

Theorem 5.5 *Let g be Lebesgue* (i.e. $\mathcal{L}^2$)*-measurable on $\mathbb{R}^2$ and assume that the iterated improper Riemann integrals*

$$\int_{-\infty}^\infty \int_{-\infty}^\infty g(x,y)dxdy \qquad \textit{and} \qquad \int_{-\infty}^\infty \int_{-\infty}^\infty g(x,y)dydx$$

exist and are finite. If one of the integrals

$$\int_{-\infty}^\infty \int_{-\infty}^\infty |g(x,y)|dxdy \qquad \textit{and} \qquad \int_{-\infty}^\infty \int_{-\infty}^\infty |g(x,y)|dydx \tag{6}$$

is finite, then the integrals of (3) *are equal.*

Proof Not given. ■

Note how we can check either of the conditions in (6). Often one of there iterated integrals is easier to evaluate than the other.