4 Integration

4.1 Integration of non-negative simple functions

Throughout we are in a measure space $(X, \mathcal{F}, \mu)$.

Definition Let s be a non-negative $\mathcal{F}$-measurable simple function so that

$$s = \sum_{i=1}^{N} a_i \chi_{A_i},$$

with disjoint $\mathcal{F}$-measurable sets $A_i, \bigcup_{i=1}^{N} A_i = X$ and $a_i \geq 0$. For any $E \in \mathcal{F}$ define the *integral of f over E* to be

$$I_E(s) = \sum_{i=1}^{N} a_i \mu(A_i \cap E),$$

with the convention that if $a_i = 0$ and $\mu(A_i \cap E) = +\infty$ then $0 \times (+\infty) = 0$. (So the area under $s \equiv 0$ on $\mathbb{R}$ is zero.)

Example 13 Consider $([0,1], \mathcal{L}, \mu)$. Define

$$f(x) = \begin{cases} 1 & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ irrational.} \end{cases}$$

This is a simple function with $A_1 = \mathbb{Q} \cap [0,1] \in \mathcal{L}$ and A_0 the set of irrationals in $[0,1]$ which, as the complement of A_1, is in $\mathcal{L}$. Thus f is measurable and

$$\begin{aligned} I_{[0,1]}(f) &= 1\mu(\mathbb{Q} \cap [0,1]) + 0\mu(\mathbb{Q}^c \cap [0,1]) \\ &= 0, \end{aligned}$$

since the Lebesgue measure of a countable set is zero.

Lemma 4.1

If $E_1 \subseteq E_2 \subseteq E_3 \ldots$ are in $\mathcal{F}$ and $E = \bigcup_{n=1}^{\infty} E_n$ then

$$\lim_{n \to \infty} \mu(E_n) = \mu(E).$$

(We say that we have an *increasing* sequence of sets.)

Proof

If there exists an n such that $\mu(E_n) = +\infty$ then $E_n \subseteq E$ implies $\mu(E) = +\infty$ and the result follows.

So assume that $\mu(E_n) < +\infty$ for all $n \geq 1$. Then

$$E = E_1 \cup \bigcup_{n=2}^{\infty} (E_n \setminus E_{n-1})$$

is a disjoint union. Note that $E_{n-1} \subseteq E_n$ implies $E_n = (E_n \setminus E_{n-1}) \cup E_{n-1}$, a disjoint union. So $\mu(E_n) = \mu(E_n \setminus E_{n-1}) + \mu(E_{n-1})$. Because the measures are finite we can rearrange as $\mu(E_n \setminus E_{n-1}) = \mu(E_n) - \mu(E_{n-1})$. So

$$\begin{aligned}
\mu(E) &= \mu(E_1) + \sum_{n=2}^{\infty} \mu(E_n \setminus E_{n-1}) \\
&= \mu(E_1) + \lim_{N\to\infty} \sum_{n=1}^{N} (\mu(E_n) - \mu(E_{n-1})) \\
&\qquad\qquad \text{(by definition of infinite sum)} \\
&= \lim_{N\to\infty} \mu(E_N).
\end{aligned}$$

■

Theorem 4.2

Let s and t be two simple non-negative $\mathcal{F}$-measurable functions on $(X, \mathcal{F}, \mu)$ and $E, F \in \mathcal{F}$. Then

(i) $I_E(cs) = cI_E(s)$ *for all* $c \in \mathbb{R}$,

(ii) $I_E(s + t) = I_E(s) + I_E(t)$,

(iii)*If* $s \leq t$ *on* E *then* $I_E(s) \leq I_E(t)$,

(iv) *If* $F \subseteq E$ *then* $I_F(s) \leq I_E(s)$,

(v) *If* $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$ *and* $E = \bigcup_{k=1}^{\infty} E_k$ *then* $\lim_{k\to\infty} I_{E_k}(s) = I_E(s)$.

Proof (Proofs of all parts will be omitted from lectures and left to students. the idea is to write out the simple functions for both s and t in terms of common sets C_{ij} as in the proof of Lemma 3.7.)

As in Lemma 3.7 write

$$s = \sum_{i=1}^{M} a_i \chi_{A_i} = \sum_{i=1}^{M} \sum_{j=1}^{N} a_i \chi_{C_{ij}}$$

and

$$t = \sum_{j=1}^{N} b_j \chi_{B_j} = \sum_{i=1}^{M} \sum_{j=1}^{N} b_j \chi_{C_{ij}}$$

with $C_{ij} = A_i \cap B_j \in \mathcal{F}$.

*(i) Note that $cs = \sum_{i=1}^{M} ca_i \chi_{A_i}$ and so

$$
\begin{aligned}
I_E(cs) &= \sum_{i=1}^{M} ca_i \mu(A_i) \\
&= c \sum_{i=1}^{M} a_i \mu(A_i) = cI_E(s).
\end{aligned}
$$

*(ii) Then $s + t = \sum_{i=1}^{M} \sum_{j=1}^{N} (a_i + b_j) \chi_{C_{ij}}$. So

$$
\begin{aligned}
I_E(s+t) &= \sum_{i=1}^{M} \sum_{j=1}^{N} (a_i + b_j) \mu(C_{ij} \cap E) \\
&= \sum_{i=1}^{M} \sum_{j=1}^{N} a_i \mu(C_{ij} \cap E) + \sum_{i=1}^{M} \sum_{j=1}^{N} b_j \mu(C_{ij} \cap E) \\
&= \sum_{i=1}^{M} a_i \mu \left(\bigcup_{j=1}^{N} (C_{ij} \cap E) \right) + \sum_{j=1}^{N} b_j \mu \left(\bigcup_{i=1}^{M} (C_{ij} \cap E) \right) \\
&= \sum_{i=1}^{M} a_i \mu(A_i \cap E) + \sum_{j=1}^{N} b_j \mu(B_j \cap E) \\
&= I_E(s) + I_E(t).
\end{aligned}
$$

*(iii) Given any $1 \leq i \leq M, 1 \leq j \leq N$ for which $C_{ij} \cap E \neq \phi$ we have for any $x \in C_{ij} \cap E$ that $a_i = s(x) \leq t(x) = b_j$ so

$$
\begin{aligned}
I_E(s) &= \sum_{i=1}^{M} \sum_{j=1}^{N} a_i \mu(C_{ij} \cap E) \\
&\leq \sum_{i=1}^{M} \sum_{j=1}^{N} b_j \mu(C_{ij} \cap E) \\
&= I_E(t).
\end{aligned}
$$

*(iv) By monotonicity of μ we have

$$\begin{aligned} I_F(s) &= \sum_{i=1}^{M} a_i \mu(A_i \cap F) \\ &\leq \sum_{i=1}^{M} a_i \mu(A_i \cap E) \\ &= I_E(s). \end{aligned}$$

*(v) From Lemma 4.1 we know that if we have $E_1 \subseteq E_2 \subseteq E_3 \subseteq ...$ and $E = \bigcup_{k=1}^{\infty} E_k$ then $\lim_{k\to\infty} \mu(E_k) = \mu(E)$. Thus

$$\begin{aligned} \lim_{k\to\infty} I_{E_k}(s) &= \lim_{k\to\infty} \sum_{i=1}^{M} a_i \mu(A_i \cap E_k) \\ &= \sum_{i=1}^{M} a_i \lim_{k\to\infty} \mu(A_i \cap E_k) \\ &= \sum_{i=1}^{M} a_i \mu(A_i \cap E) \qquad \text{by Lemma 4.1,} \\ &= I_E(s). \end{aligned}$$

■

4.2 Integration of non-negative measurable functions.

Definition If $f : X \to \mathbb{R}^+$ is a non-negative $\mathcal{F}$-measurable function, $E \in \mathcal{F}$, then the *integral of* f *over* E is

$$\int_E f d\mu = \sup \{ I_E(s) : s \text{ a simple } \mathcal{F}\text{-measurable function, } 0 \leq s \leq f \}.$$

Of course, if $E \neq X$ we need only that f is defined on some domain containing E.

Let $\mathcal{I}(f, E)$ denote the set

$$\{ I_E(s) : s \text{ a simple } \mathcal{F}\text{-measurable function, } 0 \leq s \leq f \}$$

so the integral equals $\sup \mathcal{I}(f, E)$.

Note The integral exists for **all** non-negative $\mathcal{F}$-measurable functions though it might be infinite.

If $\int_E f d\mu = \infty$ we say the integral is *defined*.

If $\int_E f d\mu < \infty$ we say that f is $\mu-$*integrable* or *summable* on E.

Proposition 4.3

For a non-negative, $\mathcal{F}$-measurable simple function, t, we have $\int_E t d\mu = I_E(t)$.

Proof

Given any simple $\mathcal{F}$-measurable function, $0 \le s \le t$ we have $I_E(s) \le I_E(t)$ by Theorem 4.2(iii). So $I_E(t)$ is **an** upper bound for $\mathcal{I}(t, E)$ for which $\int_E t d\mu$ is the **least** of all upper bounds. Hence $\int_E t d\mu \le I_E(t)$.

Also, $\int_E t d\mu \ge I_E(s)$ for all simple $\mathcal{F}$-measurable function, $0 \le s \le t$, and so is greater than $I_E(s)$ for any particular s, namely $s = t$. Hence $\int_E t d\mu \ge I_E(t)$.

Thus $\int_E t d\mu = I_E(t)$. ■

Example 14 If $f \equiv k$, a constant, then $\int_E f d\mu = I_E(f) = k\mu(E)$.

Theorem 4.4 *Throughout, all sets are in $\mathcal{F}$ and all functions are non-negative and $\mathcal{F}$-measurable.*

(i) *For all $c \ge 0$,*

$$\int_E cf d\mu = c \int_E f d\mu, \tag{15}$$

(ii) *If $0 \le g \le h$ on E then*

$$\int_E g d\mu \le \int_E h d\mu,$$

(iii) *If $E_1 \subseteq E_2$ and $f \ge 0$ then*

$$\int_{E_1} f d\mu \le \int_{E_2} f d\mu.$$

Proof

(i) If $c = 0$ then the right hand side of (15) is 0 as is the left hand side by Example 14.

Assume $c > 0$.

If $0 \le s \le cf$ is a simple $\mathcal{F}$-measurable function then so is $0 \le \frac{1}{c}s \le f$. Thus

$$\int_E f d\mu \ge I_E\left(\frac{1}{c}s\right) = \frac{1}{c} I_E(s)$$

by Theorem 4.2(i). Hence $c\int_E f d\mu$ is **an** upper bound for $\mathcal{I}(cf, E)$ for which $\int_E cf d\mu$ is the **least** upper bound. Thus $c\int_E f d\mu \ge \int_E cf d\mu$.

Starting with the observation that if $0 \le s \le f$ is a simple $\mathcal{F}$-measurable function then so is $0 \le cs \le cf$ we obtain

$$\begin{aligned}
\int_E (cf)\, d\mu &\ge I_E(cs) && \text{by the definition of } \int_E \\
&= cI_E(s) && \text{by Theorem 4.2(i).}
\end{aligned}$$

Hence $\frac{1}{c}\int_E (cf) d\mu$ is **an** upper bound for $\mathcal{I}(f, E)$ for which $\int_E f d\mu$ is the **least** upper bound. Hence $\frac{1}{c}\int_E (cf) d\mu \ge \int_E f d\mu$, that is, $\int_E cf d\mu \ge c\int_E f d\mu$.

Combining both inequalities gives our result.

(ii) Let $0 \le s \le g$ be a simple, $\mathcal{F}$-measurable function. Then since $g \le h$ we trivially have $0 \le s \le h$ in which case $I_E(s) \le \int_E h d\mu$ by the definition of integral $\int_E$. Thus $\int_E h d\mu$ is **an** upper bound for $\mathcal{I}(g, E)$. As in (i) we get $\int_E h d\mu \ge \int_E g d\mu$.

(iii) Let $0 \le s \le f$ be a simple, $\mathcal{F}$-measurable function. Then

$$\begin{aligned}
I_{E_1}(s) &\le I_{E_2}(s) && \text{by Theorem 4.2(iii)} \\
&\le \int_{E_2} f d\mu && \text{by the definition of } \int_{E_2}.
\end{aligned}$$

So $\int_{E_2} f d\mu$ is **an** upper bound for $\mathcal{I}(f, E_1)$ and so is greater than the least of all upper bounds. Hence $\int_{E_2} f d\mu \ge \int_{E_1} f d\mu$. ■

Lemma 4.5

Assume $E \in \mathcal{F}$, $f \ge 0$ is $\mathcal{F}$-measurable and $\int_E f d\mu < \infty$. Set

$$A = \{x \in E : f(x) = +\infty\}.$$

Then $A \in \mathcal{F}$ and $\mu(A) = 0$.

Proof

Since f is $\mathcal{F}$-measurable then $f^{-1}(\{\infty\}) \in \mathcal{F}$ and so $A = E \cap f^{-1}(\{\infty\}) \in \mathcal{F}$. Define

$$s_n(x) = \begin{cases} n & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Since $A \in \mathcal{F}$ we deduce that s_n is an $\mathcal{F}$-measurable simple function. Also $s_n \le f$ and so

$$\begin{aligned}
n\mu(A) &= I_E(s_n) && \text{by definition of } I_E \\
&\le \textstyle\int_E f d\mu && \text{by definition of } \textstyle\int_E \\
&< \infty && \text{by assumption.}
\end{aligned}$$

True for all $n \geq 1$ means that $\mu(A) = 0$. ■

Lemma 4.6

If f is $\mathcal{F}$-measurable and non-negative on $E \in \mathcal{F}$ and $\mu(E) = 0$ then $\int_E f d\mu = 0$.

Proof

Let $0 \leq s \leq f$ be a simple, $\mathcal{F}$-measurable function. So $s = \sum_{n=1}^{N} a_n \chi_{A_n}$ for some $a_n \geq 0, A_n \in \mathcal{F}$. Then $I_E(s) = \sum_{n=1}^{N} a_n \mu(A_n \cap E)$. But μ is monotone which means that $\mu(A_n \cap E) \leq \mu(E) = 0$ for all n and so $I_E(s) = 0$ for all such simple functions. Hence $\mathcal{I}(f, E) = \{0\}$ and so $\int_E f d\mu = \sup \mathcal{I}(f, E) = 0$. ■

Lemma 4.7 *If $g \geq 0$ and $\int_E g d\mu = 0$ then*

$$\mu\{x \in E : g(x) > 0\} = 0.$$

Proof Let $A = \{x \in E : g(x) > 0\}$ and $A_n = \{x \in E : g(x) > \frac{1}{n}\}$. Then the sets $A_n = E \cap \{x : g(x) > \frac{1}{n}\} \in \mathcal{F}$ satisfy $A_1 \subseteq A_2 \subseteq A_3 \subseteq ...$ with $A = \bigcup_{n=1}^{\infty} A_n$. By lemma 4.1 $\mu(A) = \lim_{n\to\infty} \mu(A_n)$. Using

$$s_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in A_n \\ 0 & \text{otherwise,} \end{cases}$$

so $s_n \leq g$ on A_n we have

$$\begin{aligned} \tfrac{1}{n}\mu(A_n) &= I_{A_n}(s_n) \\ &\leq \int_{A_n} g d\mu \quad \text{by the definition of } \int_{A_n} \\ &\leq \int_E g d\mu \quad \text{Thereom 4.4(iii)} \\ &= 0 \quad \text{by assumption.} \end{aligned}$$

So $\mu(A_n) = 0$ for all n and hence $\mu(A) = 0$. ■

Definition If a property P holds on all points in $E \setminus A$ for some set A with $\mu(A) = 0$ we say that P holds *almost everywhere* (μ) *on* E, written as *a.e.*(μ) *on* E.

(*It might be that P holds on some of the points of A or that the set of points on which P does not hold is non-measurable. This is immaterial. But if μ is a complete measure, such as the Lebesgue-Steiltje's measure μ_F, then the situation is simpler. Assume that a property P holds *a.e.*(μ) *on* E. The definition says that the set of points, D say, on which P does not hold can be covered by a set of measure zero, i.e. there exists $A : D \subseteq A$ and $\mu(A) = 0$. Yet if μ is complete then D will be measurable of measure zero.

In this section we are not assuming that μ is complete.)

So, for example, Lemma 4.7 can be restated as

Lemma 4.8

If $g \geq 0$ *and* $\int_E g d\mu = 0$ *then* $g = 0$ *a.e.*(μ) *on* E.

We can extend Theorem 4.4(ii) as follows.

Theorem 4.9 *If* $g, h : X \to \mathbb{R}^+$ *are* $\mathcal{F}$*-measurable functions and* $g \leq h$ *a.e.*(μ) *then*

$$\int_E g d\mu \leq \int_E h d\mu.$$

Proof

By assumption there exists a set $D \subseteq E$, of measure zero, such that for all $x \in E \setminus D$ we have $g(x) \leq h(x)$. Let $0 \leq s \leq g$ be a simple, $\mathcal{F}$-measurable function, written as

$$s = \sum_{i=1}^{N} a_i \chi_{A_i}, \qquad \text{with } \bigcup_{i=1}^{N} A_i = E.$$

The problem here is that we may well not have $s \leq h$. Define

$$\begin{aligned} s^*(x) &= \begin{cases} s(x) & \text{if } x \notin D \\ 0 & \text{if } x \in D \end{cases} \\ &= \sum_{i=1}^{N} a_i \chi_{A_i \cap D^c} \end{aligned}$$

which is still a simple, $\mathcal{F}$-measurable function. Then for $x \in E \setminus D$ we have $s^*(x) = s(x) \leq g(x) \leq h(x)$, while for $x \in D$ we have $s^*(x) = 0 \leq h(x)$. Thus $s^*(x) \leq h(x)$ for all $x \in E$.

Note that $A_i = (A_i \cap D^c) \cup (A_i \cap D)$, a disjoint union in which case $\mu(A_i) = \mu(A_i \cap D^c) + \mu(A_i \cap D) = \mu(A_i)$. But $A_i \cap D \subseteq D$ and so $\mu(A_i \cap D) \leq \mu(D) = 0$. Thus $\mu(A_i) = \mu(A_i \cap D^c)$. Hence

$$\begin{aligned} I_E(s^*) &= \sum_{i=1}^{N} a_i \mu(A_i \cap D^c) \\ &= \sum_{i=1}^{N} a_i \mu(A_i) \\ &= I_E(s). \end{aligned}$$

So $I_E(s) = I_E(s^*) \leq \int_E h d\mu$ by the definition of integral $\int_E$. Thus $\int_E h d\mu$ is **an** upper bound for $\mathcal{I}(g, E)$ while $\int_E g d\mu$ is the **least** of all upper bounds for $\mathcal{I}(g, E)$. Hence $\int_E h d\mu \geq \int_E g d\mu$. ■

Corollary 4.10

If $g, h : X \to \mathbb{R}^+$ are $\mathcal{F}$-measurable with $g = h$ a.e (μ) on E then

$$\int_E g d\mu = \int_E h d\mu.$$

Proof

By assumption there exists a set $D \subseteq E$ of measure zero such that for all $x \in E \setminus D$ we have $g(x) = h(x)$. In particular, for these x we have $g(x) \leq h(x)$ and $h(x) \leq g(x)$. So $g \leq h$ a.e. (μ) on E and $h \leq g$ a.e. (μ) on E. Hence the result follows from two applications of Theorem 4.9. ■

So, a function may have its values altered on a set of measure zero without altering the value of its integral. In particular, by Lemma 4.5 we may assume that a non-negative integrable function is finite valued.

Example 15 (c.f. Example 13) On $([0, 1], \mathcal{L}, \mu)$ the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ irrational} \end{cases}$$

is 0 a.e.(μ) on $[0, 1]$. So

$$\int_{[0,1]} f d\mu = \int_{[0,1]} 0 d\mu = 0.$$