3.2 Simple Functions

Definition A function $f : X \to \mathbb{R}$ is *simple* if it takes only a finite number of different values.

Note these values must be finite. Writing them as $a_i, 1 \leq i \leq N$, and letting $A_i = \{x \in X : f(x) = a_i\}$, we can write

$$f = \sum_{i=1}^{N} a_i \chi_{A_i}$$

where χ_A is the characteristic function of A, that is, $\chi_A(x) = 1$ if $x \in A$, and 0 otherwise.

Lemma 3.7

The simple functions are closed under addition and multiplication.

Proof

Let $s = \sum_{i=1}^{M} a_i \chi_{A_i}$ and $t = \sum_{j=1}^{N} b_j \chi_{B_j}$ where $\bigcup_{i=1}^{M} A_i = \bigcup_{j=1}^{N} B_j = X$.

Define $C_{ij} = A_i \cap B_j$. Then $A_i \subseteq X = \bigcup_{j=1}^{N} B_j$ and so $A_i = A_i \cap \bigcup_{j=1}^{N} B_j$ $= \bigcup_{j=1}^{N} C_{ij}$. Similarly $B_j = \bigcup_{i=1}^{M} C_{ij}$. Since the C_{ij} are disjoint this means that

$$\chi_{A_i} = \sum_{j=1}^{N} \chi_{C_{ij}} \quad \text{and} \quad \chi_{B_j} = \sum_{i=1}^{M} \chi_{C_{ij}}.$$

Thus

$$s = \sum_{i=1}^{M} \sum_{j=1}^{N} a_i \chi_{C_{ij}} \quad \text{and} \quad t = \sum_{i=1}^{M} \sum_{j=1}^{N} b_j \chi_{C_{ij}}.$$

Hence

$$s + t = \sum_{i=1}^{M} \sum_{j=1}^{N} (a_i + b_j) \chi_{C_{ij}} \quad \text{and} \quad st = \sum_{i=1}^{M} \sum_{j=1}^{N} a_i b_j \chi_{C_{ij}}$$

are simple functions. ■

Let $\mathcal{F}$ be a σ-field on X. Assume that for a simple function f we have $A_i \in \mathcal{F}$ for all i. Then

$$\{x : f(x) > c\} = \bigcup_{a_i > c} A_i \in \mathcal{F}$$

for all $c \in \mathbb{R}$. Hence f is $\mathcal{F}$-measurable. Conversely assume that f is $\mathcal{F}$-measurable. Order the values attained by f as $a_1 < a_2 < ... < a_N$. Given

$1 \leq j \leq N$ choose $a_{j-1} < c_1 < a_j < c_2 < a_{j+1}$. (If $j = 1$ or N part of this requirement is empty.) Then

$$\begin{aligned} A_j &= \left(\bigcup_{a_i > c_1} A_i \right) \setminus \left(\bigcup_{a_i > c_2} A_i \right) \\ &= \{x : f(x) > c_1\} \setminus \{x : f(x) > c_2\} \\ &\in \mathcal{F}. \end{aligned}$$

Hence

Lemma 3.8

If $f : (X, \mathcal{F}) \to \mathbb{R}$ is a simple function then f is $\mathcal{F}$-measurable if, and only if, $A_i \in \mathcal{F}$ for all $1 \leq i \leq N$.

■

Corollary 3.9

The simple $\mathcal{F}$-measurable functions are closed under addition and multiplication.

Proof

Simply note in the proof of Lemma 3.7 that since A_i and B_j are in $\mathcal{F}$ then $C_{ij} \in \mathcal{F}$. ■

Note If s is a simple function and $g : \mathbb{R} \to \mathbb{R}$ is any function whose domain contains the values of s then $g \circ s$ (defined by $(g \circ s)(x) = g(s(x))$) is simple. In fact

$$g \circ s = \sum_{i=1}^{N} g(a_i) \chi_{A_i} = \sum_{j=1}^{M} b_j \chi_{B_i}$$

for some $M \leq N$ and where $B_j = \bigcup_{g(a_i)=b_j} A_i$. Also if s is $\mathcal{F}$-measurable then $g \circ s$ is too.

The next result is very important.

Theorem 3.10

Let f be a non-negative $\mathcal{F}$-measurable function. Then there exist a sequence of simple $\mathcal{F}$-measurable functions s_n such that $0 \leq s_1 \leq ... \leq s_n \leq s_{n+1} \leq ...$ and $\lim_{n\to\infty} s_n = f$.

Proof

We partition the range of f using the points in $\mathcal{D}_n = \left\{ \frac{\nu}{2^n} : 0 \leq \nu \leq n2^n \right\}$. Importantly, though trivial, we have $\mathcal{D}_n \subseteq \mathcal{D}_{n+1}$.

Define $s_n(x) = \max\{\gamma \in \mathcal{D}_n, \gamma \leq f(x)\}$.

Then $\mathcal{D}_n \subseteq \mathcal{D}_{n+1}$ means that for any given x,

$$\{\gamma \in \mathcal{D}_n, \gamma \leq f(x)\} \subseteq \{\gamma \in \mathcal{D}_{n+1}, \gamma \leq f(x)\}$$

and so

$$\begin{aligned} s_n(x) &= \max\{\gamma \in \mathcal{D}_n, \gamma \leq f(x)\} \\ &\leq \max\{\gamma \in \mathcal{D}_{n+1}, \gamma \leq f(x)\} \\ &= s_{n+1}(x). \end{aligned}$$

True for all x means that $s_n \leq s_{n+1}$ as required. It also means that $\lim_{n\to\infty} s_n(x)$ exists (using the extended definition of limit if necessary).

Look first at those x for which $f(x)$ is finite. Then for all n for which $n \geq f(x)$ we have $s_n(x) = \nu/2^n$ for the $0 \leq \nu \leq n2^n$ satisfying

$$\frac{\nu}{2^n} \leq f(x) < \frac{\nu+1}{2^n}, \qquad \text{that is} \qquad s_n(x) \leq f(x) < s_n(x) + \frac{1}{2^n}.$$

Hence, by the sandwich rule, $\lim_{n\to\infty} s_n(x) = f(x)$.

Look now at x such that $f(x) = +\infty$. Then $s_n(x) = n$ for all n. Hence $\lim_{n\to\infty} s_n(x) = +\infty$ by the extended definition of limit. Thus $\lim_{n\to\infty} s_n(x) = f(x)$.

True for all x means $\lim_{n\to\infty} s_n = f$.

Finally

$$s_n(x) = \sum_{0\leq\nu\leq n2^n} \frac{\nu}{2^n} \chi_{A_{\nu,n}}(x)$$

where

$$\begin{aligned} A_{\nu,n} &= \left\{x : \frac{\nu}{2^n} \leq f(x) < \frac{\nu+1}{2^n}\right\} \\ &= \left\{x : f(x) < \frac{\nu+1}{2^n}\right\} \setminus \left\{x : f(x) < \frac{\nu}{2^n}\right\} \end{aligned}$$

for $\nu \leq n2^n - 1$ while

$$A_{n2^n,n} = \{x : f(x) \geq n\}.$$

In all cases the sets $A_{\nu,n} \in \mathcal{F}$. So the s_n are simple $\mathcal{F}$-measurable functions. ∎

Combining Theorems 3.10 and 3.6 we see that a function $f : (X, \mathcal{F}) \to \mathbb{R}^+$ is $\mathcal{F}$-measurable if, and only if, there exists an increasing sequence of simple, $\mathcal{F}$-measurable functions converging to f.

Corollary 3.11

If $f : (X, \mathcal{F}) \to \mathbb{R}^*$ is $\mathcal{F}$-measurable then it is the limit of a sequence of simple $\mathcal{F}$-measurable functions.

Proof

As in the proof of Theorem 3.4(viii) we can write $f = f^+ - f^-$ where f^+ and f^- are non-negative $\mathcal{F}$-measurable functions. So by Theorem 3.10 we can find sequences of simple, $\mathcal{F}$-measurable functions $s_n \to f^+$ and $t_n \to f^-$ in which case $\{s_n - t_n\}_{n\geq 1}$ is the required sequence of simple functions (using Lemma 3.7) converging to f. ■

Corollary 3.12

If $f : (X, \mathcal{F}) \to \mathbb{R}^*$ is $\mathcal{F}$-measurable and $g : \mathbb{R} \to \mathbb{R}$ a continuous function whose domain contains the values of f then the composition function $g \circ f$ is $\mathcal{F}$-measurable.

Proof

By Corollary 3.11 we can find a sequence of simple, $\mathcal{F}$-measurable functions $s_n \to f$. By an earlier note the functions $g \circ s_n$ are simple and still $\mathcal{F}$-measurable for all n. Then

$$\begin{aligned} \lim_{n\to\infty} g(s_n(x)) &= g(\lim_{n\to\infty} s_n(x)) \quad \text{since } g \text{ is continuous,} \\ &= g(f(x)) \\ &= (g \circ f)(x) \end{aligned}$$

for all $x \in X$, i.e. $g \circ f = \lim_{n\to\infty} g \circ s_n$. Hence, by Theorem 3.6, $g \circ f$ is $\mathcal{F}$-measurable. ■

Example 12 If $f : (X, \mathcal{F}) \to \mathbb{R}^+$ is $\mathcal{F}$-measurable then $\sin f$, $\exp(f)$ and $\log f$ are also $\mathcal{F}$-measurable on the set of x on which they are defined.