Appendix to Notes 8 (b)

Spaces of Integrable functions

Definition On a vector space V over $\mathbb{R}$ a *norm* is a function $||.|| : V \to \mathbb{R}$ satisfying, for all $x, y \in V$,

(i) $||x|| \geq 0$ and equals 0 if, and only if, $x = 0$,
(ii) $||ax|| = |a|||x||$ for all real numbers a,
(iii) $||x + y|| \leq ||x|| + ||y||$.

Define

$$L^1(\mu) = \left\{ f : \int_X |f| d\mu < \infty \right\} \quad \text{with} \quad ||f||_1 = \int_X |f| d\mu$$

and, in general,

$$L^p(\mu) = \left\{ f : \int_X |f|^p d\mu < \infty \right\} \quad \text{with} \quad ||f||_p = \left(\int_X |f|^p d\mu \right)^{1/p},$$

as subsets of all measurable functions.

We have to first check that these spaces are vector spaces over $\mathbb{R}$, only then can we check if the $||.||_1$ and $||.||_p$ are norms. So we have to check that if f, g are from our set and $a, b \in \mathbb{R}$ then $af + bg$ is in our set. That $L^1(\mu)$ is a vector space follows from

$$|af + bg| \leq |a||f| + |b||g|.$$

In general we use

$$\begin{aligned} |af + bg|^p &\leq (|a||f| + |b||g|)^p \\ &\leq \begin{cases} (2|a||f|)^p & \text{if} \quad |a||f| \geq |b||g| \\ (2|b||g|)^p & \text{if} \quad |b||g| \geq |a||f| \end{cases} \\ &= 2^p \max((|a||f|)^p, (|b||g|)^p) \\ &\leq 2^p((|a||f|)^p + (|b||g|)^p). \end{aligned}$$

So if $f, g \in L^p(\mu)$ then

$$\int_X |af + bg|^p d\mu \leq 2^p \left(|a|^p \int_X |f|^p d\mu + |b|^p \int_X |g|^p d\mu \right) < \infty$$

and so $af + bg \in L^p(\mu)$.

In fact we have to look upon $L^p(\mu)$ as **spaces of equivalence classes** where the relation is given by $f \sim g$ when $f = g$ a.e.(μ) on X. Of course, if $||f||_1 = 0$ or $||f||_p = 0$ then $f = 0$ a.e.(μ), but the set of all such f is an equivalence class and thus just one element in this interpretation of $L^1(\mu)$ and $L^p(\mu)$. This was exactly what was reuired for a norm function. So it is not meaningful, when considering a "function" from $L^1(\mu)$ or $L^p(\mu)$, to ask for the value of that function at a given point. We only ever deal with a representative of an equivalence class and we can change the representative on a set of measure zero if and when necessary. Part (ii) of the definition of a norm function is obviously satisfied for $||.||_1$ and $||.||_p$ so we just have to check part (iii).

Lemma 1 *For reals $a, b > 0$ and $0 < t < 1$ we have*

$$a^t b^{1-t} \leq ta + (1-t)b.$$

Proof

If $w > 1$ and $t < 1$ then

$$\begin{aligned} w^t - 1^t &= \int_1^w d(x^t) = t\int_1^w x^{t-1}dx \\ &\leq t\int_1^w dx = t(w-1). \end{aligned} \qquad (1)$$

If $a > b$ set $w = a/b$ to get

$$\left(\frac{a}{b}\right)^t - 1 \leq t\left(\frac{a}{b} - 1\right),$$

so

$$a^t b^{1-t} \leq ta + (1-t)b.$$

If $b \geq a$ use $w^{1-t} - 1 \leq (1-t)(w-1)$ which follows from (1) on replacing t by $1-t$, valid since $1-t < 1$. ■

Lemma 2 *Let $1 \leq p < \infty$ and set $1/q = 1 - 1/p$. If $f \in L^p(\mu)$ and $g \in L^q(\mu)$ then $fg \in L^1(\mu)$ and*

$$||fg||_1 = \int_X |f||g|d\mu \leq ||f||_p||g||_q$$

Proof Apply Lemma 1 with

$$a = \frac{|f|^p}{||f||_p^p} \text{ and } b = \frac{|g|^q}{||g||_q^q}$$

and $t = 1/p$. Then

$$\frac{|f|}{||f||_p}\frac{|g|}{||g||_q} \le \frac{|f|^p}{p||f||_p^p} + \frac{|g|^q}{q||g||_q^q}.$$

Integrate over X to get

$$\begin{aligned}
\frac{1}{||f||_p||g||_q}\int_X |f||g|d\mu &\le \frac{1}{p||f||_p^p}\int_X |f|^p d\mu + \frac{1}{q||g||_q^q}\int_X |g|^q d\mu \\
&= \frac{||f||_p^p}{p||f||_p^p} + \frac{||g||_q^q}{q||g||_q^q} \\
&= \frac{1}{p} + \frac{1}{q} = 1
\end{aligned}$$

■

Lemma 3 *If* $f, g \in L^p(\mu)$ *then*

$$||f+g||_p \le ||f||_p + ||g||_p.$$

Proof By first applying the triangle inequality

$$\begin{aligned}
\int_X |f+g|^p d\mu &\le \int_X |f||f+g|^{p-1}d\mu + \int_X |g||f+g|^{p-1}d\mu \\
&\le \left(\int_X |f|^p d\mu\right)^{1/p}\left(\int_X \left(|f+g|^{p-1}\right)^q d\mu\right)^{1/q} \\
&\quad + \left(\int_X |g|^p d\mu\right)^{1/p}\left(\int_X \left(|f+g|^{p-1}\right)^q d\mu\right)^{1/q} \\
&= (||f||_p + ||g||_p)\left(\int_X |f+g|^p d\mu\right)^{1/q} \quad \text{since } q(p-1) = p.
\end{aligned}$$

Hence, on rearranging,

$$\left(\int_X |f+g|^p d\mu\right)^{1-1/q} \le ||f||_p + ||g||_p,$$

and the left hand side equals $||f+g||_p$ since $1 - 1/q = 1/p$. ■

Definition Let $(V, ||.||)$ be a normed space. A *Cauchy sequence* $\{x_n\}$ satisfies

$$\forall \varepsilon > 0, \exists N \ge 1, \forall m, n \ge N, |x_m - x_n| < \varepsilon.$$

We say that $(V, ||.||)$ is *complete* if every Cauchy sequence is convergent.

It can be shown that $\mathbb{R}$ with the usual distance function is complete.

Theorem 1

$$L^1(\mu) \textit{ is complete.}$$

Proof

Let $\{f_n\}$ be a Cauchy sequence in $L^1(\mu)$. We need show that $\lim_{n\to\infty} f_n$ exists and, calling it f, that $f \in L^1(\mu)$ and $||f_n - f||_1 \to 0$ as $n \to \infty$.

By definition of a Cauchy sequence we have that, for all $\varepsilon > 0$ there exists N such that $||f_m - f_n||_1 < \varepsilon$ for all $m, n \geq N$. We will apply this with a sequence of ε_k such that $\sum_k \varepsilon_k < \infty$. In particular , $\varepsilon_k = 1/4^k$. So there exists N_k such that $||f_m - f_n||_1 < 1/4^k$ for all $m, n \geq N_k$. And we can ensure that $N_1 < N_2 < N_3 < ...$.

Set $g_k = f_{N_k}$, so $||g_m - g_n||_1 < 1/4^n$ for all $m \geq n$. In particular

$$\int_X |g_m - g_n| d\mu < 1/4^n.$$

Thus $g_m - g_n$ is small on average. Though it can be large it cannot be large for too many x. (This is the idea behind the Chebychev inequality.)

Claim $\lim_{n\to\infty} g_n$ exists a.e.(μ).

The idea is to write $h_k = g_k - g_{k-1}$ with $h_1 = g_1$ so that $g_k = \sum_{j=1}^k h_j$. The hope then is to find a sequence of sets $E_1 \supseteq E_2 \supseteq E_3 \supseteq ...$ such that outside each E_n the series $\sum_{j=1}^\infty h_j$converges. The proof of convergence is by the comparison test. We will find that the smaller we take E_n the larger we have to take the comparing series. (This just represents the fact that the convergence of the series need not be uniform across X.) Nonetheless we hope that $\mu\left(\bigcap_{n\geq 1} E_n\right) = 0$ so that we get convergence a.e.(μ).

Let $n \geq 1$ be given. Define

$$\begin{aligned} E_n &= \left\{x : |h_j(x)| > \frac{n}{2^j} \text{ for some } j \geq 1\right\} \\ &= \left\{x : |g_j(x) - g_{j-1}(x)| > \frac{n}{2^j} \text{ for some } j \geq 1\right\} \\ &= \bigcup_{j\geq 1} \left\{x : |g_j(x) - g_{j-1}(x)| > \frac{n}{2^j}\right\} \\ &= \bigcup_{j\geq 1} E_{n,j}, \text{ say.} \end{aligned}$$

Then $\mu(E_n) \leq \sum_{j\geq 1} \mu(E_{n,j})$ and

$$\begin{aligned}\frac{n}{2^j}\mu(E_{n,j}) &\le \int_{E_{n,j}} |g_j - g_{j-1}| d\mu \\ &\le \int_X |g_j - g_{j-1}| d\mu \\ &< \frac{1}{4^{j-1}}.\end{aligned}$$

Hence $\mu(E_{n,j}) \le 4/2^j n$ and so $\mu(E_n) \le \sum_{j\ge1} 4/2^j n = 4/n$. Thus $\mu((\bigcap_{n\ge1} E_n) = 0$.

So, for all $x \notin E_n$ we have $|h_j(x)| \le n/2^j$ for all j in which case $|g_k(x) - g_l(x)| < n/2^l$ for all $k \ge l$. The sequence $\{g_k(x)\}_k$ is a Cauchy sequence in $\mathbb{R}$ and so converges. Hence for all $x \notin \bigcap_{n\ge1} E_n$ we have that $\{g_k(x)\}_k$ converges, that is $\{g_k\}_k$ converges a.e. (μ) on X.

Let $E = \bigcap_{n\ge1} E_n$. Define

$$f(x) = \begin{cases} \lim_{k\to\infty} g_k(x) & \text{for } x \in X \setminus E \\ 0 & \text{for } x \in E. \end{cases}$$

Then go back, and for each f_n choose a function from the same equivalence class that is zero on E (possible since $\mu(E) = 0$) and relabel as f_n. Hence $f(x) = \lim_{k\to\infty} g_k(x) = \lim_{k\to\infty} f_{N_k}(x)$ for all $x \in X$. Though we have found pointwise limit for a subsequence of $\{f_n\}$ it would be too much to expect that f would be the pointwise limit of the sequence $\{f_n\}$. Yet we should be able to show that f is the limit "on average", i.e. that $||f_n - f||_1 \to 0$ as $n \to \infty$. Before we do this we need know that we can calculate $||f_n - f||_1$, that is, we need to know that $f \in L^1(\mu)$

Define $h = \sum_{k\ge1} |h_k|$ which converges for all $x \notin E$. Define $h_k(x) = 0$ for $x \in E$ for all $k \ge 1$. So then the series converges for all x. Then, by the Monotonic Convergence Theorem,

$$\begin{aligned}\int_X h d\mu &= \sum_{k\ge1} \int_X |h_k| d\mu \\ &= \int_X |f_1| d\mu + \sum_{k\ge2} \int_X |g_k - g_{k-1}| d\mu \\ &\le ||f_1||_1 + \sum_{k\ge2} \frac{1}{4^{k-1}} \\ &= ||f_1||_1 + C, \text{ say, which is finite.}\end{aligned}$$

Hence h is integrable. Note that for each $k \ge 1$ we have

$$|g_k| = \left|\sum_{j=1}^{k} h_j\right| \leq \sum_{j=1}^{k} |h_j| \leq h$$

and so

$$|f| = \lim_{k\to\infty} |g_k| \leq h.$$

Theus we have, by Corollary 4.18, that f is integrable, i.e. $f \in L^1(\mu)$.

Next

$$|g_k - f| \leq |g_k| + |f| \leq 2h.$$

So by the dominated convergence theorem,

$$\lim_{k\to\infty} \int_X |g_k - f| d\mu = \int_X \lim_{k\to\infty} |g_k - f| d\mu = 0,$$

that is, $\lim_{k\to\infty} ||g_k - f||_1 = 0$.

Finally let $\varepsilon > 0$ be given. We are told that $\{f_n\}$ is a Cauchy sequence so we can find N such that $||f_n - f_m||_1 < \varepsilon$ for all $n, m > N$. So for $N_k > N$ we have $||f_{N_k} - f_m||_1 < \varepsilon$ that is $||g_k - f_m||_1 < \varepsilon$. Let $k \to \infty$ to deduce $||f - f_m||_1 < \varepsilon$. True for all $m > N$ means that $\lim_{m\to\infty} ||f_m - f||_1 = 0$ as required. ∎

Theorem 2

$L^p(\mu)$ is complete.

Proof We can use the method of proof above but here we give an alternative proof.

Given a Cauchy sequence $\{f_n\}$ in $L^p(\mu)$ we can find a subsequence such that

$$||f_{n_{i+1}} - f_{n_i}||_p < \frac{1}{2^i}$$

for all $i \geq 1$. Let

$$g_k = \sum_{i=1}^{k} (f_{n_{i+1}} - f_{n_i}).$$

Then, by the triangle inequality for norms,

$$
\begin{aligned}
||g_k||_p &\leq \sum_{i=1}^{k} ||f_{n_{i+1}} - f||_p \\
&\leq \sum_{i=1}^{k} \frac{1}{2^i} \leq 1
\end{aligned}
$$

Thus

$$
\begin{aligned}
||g||_p^p &= \int_X |g|^p d\mu = \int_X \lim_{k\to\infty} |g_k|^p d\mu \\
&\leq \liminf_{k\to\infty} \int_X |g_k|^p d\mu \quad \text{by Fatou's lemma} \\
&\leq 1, \quad \text{by above.}
\end{aligned}
$$

In particular, g is finite a.e.(μ) on X. Let $f = g$ where g is defined, 0 elsewhere and go back and choose f_n so that they too are zero where g is not defined.

Let $\varepsilon > 0$ be given. Then there exists N such that $||f_n - f_m||_p < \varepsilon$ for all $m, n \geq N$. Choose such an m. Then

$$
\begin{aligned}
\int_X |f - f_m|^p d\mu &= \int_X \lim_{i\to\infty} |f_{n_i} - f_m|^p d\mu \\
&\leq \liminf_{i\to\infty} \int_X |f_{n_i} - f_m|^p d\mu, \quad \text{by Fatou's lemma,} \\
&\leq \varepsilon^p.
\end{aligned}
$$

In particular, $f - f_m \in L^p(\mu)$. But we know that $f_m \in L^p(\mu)$ hence $f \in L^p(\mu)$. Also, given $\varepsilon > 0$ we have found an N such that $||f - f_m||_p \leq \varepsilon$ for all $m \geq N$. Hence $\lim_{m\to\infty} ||f - f_m||_p = 0$. ■