Appendix to Notes 7

Extended version of Monotonic Convergence Theorem

From the notes recall the following important result.

Theorem 4.11 Lebesgue's Monotone Convergence Theorem

Let $0 \leq f_1 \leq ... \leq f_n \leq f_{n+1} \leq ...$ *be an increasing sequence of non-negative* $\mathcal{F}$*-measurable functions. Let* $E \in \mathcal{F}$*. Then*

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E \lim_{n\to\infty} f_n d\mu.$$

We can try to extend this Theorem. The result is often stated under the condition that $\lim_{n\to\infty} f_n = f$ a.e. (μ) on E but this will follow from Theorem 4.11 if we simply apply Corollary 4.10. We can go further. Perhaps we only have $f_n \leq f_{n+1}$ a.e.(μ) on E. That is, there exists a set A_n with zero measure so that for all $x \in X \setminus A_n$ we have $f_n(x) \leq f_{n+1}(x)$. Let $A = \bigcup_{n=1}^{\infty} A_n$ so that, by countable sub-additivity, $\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n) = 0$. Then for all $x \in E \setminus A$ we have

$$f_1(x) \leq f_2(x) \leq f_3(x) \leq ... \,.$$

So $\lim f_n$ exists a.e.(μ). Let us suppose that f is an $\mathcal{F}$-measurable non-negative function defined on all of E such that on $E \setminus A$ we have $f = \lim f_n$ a.e.(μ). That is, there exists a set $B \subseteq E \setminus A$ of measure zero so that for all $x \in (E \setminus A) \setminus B = X \setminus (A \cup B)$ we have $f(x) = \lim_{n\to\infty} f_n(x)$.

Theorem 1

With the conditions above, and assuming that μ *is complete*

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu.$$

Proof

The inequality $f_n \leq f_{n+1}$ a.e.(μ) on E means that $\int_E f_n d\mu \leq \int_E f_{n+1} d\mu$ for all n so $L = \lim_{n\to\infty} \int_E f_n d\mu$ exists, possibly infinite.

Note that for $x \in X \setminus (A \cup B)$ we have $f_n(x) \leq \lim_{m\to\infty} f_m(x) = f(x)$ so, for every $n \geq 1$, $f_n \leq f$ a.e. (μ) on X. Thus $\int_E f_n d\mu \leq \int_E f d\mu$ for all n. Hence

$$L \leq \int_E f d\mu. \tag{1}$$

Let $0 \leq s \leq f$ be any simple $\mathcal{F}$-measurable function on E and let $0 \leq c \leq 1$. Set $E_n = \{x \in E : cs(x) \leq f_n(x)\}$. It is **not** necessarily true that

$E_n \subseteq E_{n+1}$. For instance if $x \in E_n \cap A_n$ then we will have $cs(x) \leq f_n(x)$ since $x \in E_n$ and we may have $f_n(x) > f_{n+1}(x)$ since $x \in A$. So it is possible that $cs(x) > f_{n+1}(x)$ that is, $x \notin E_{n+1}$. But certainly $E_n \cap A_n^c \subseteq E_{n+1}$, so almost all of E_n lies in E_{n+1} in that $E_n \setminus E_{n+1} \subseteq (E_n \cap A_n) \setminus E_{n+1} \subseteq A_n$, i.e. $\mu(E_n \setminus E_{n+1}) = 0$. Nonetheless, we have the following:

Lemma 1 *If* $E_1, E_2, E_2, ... \in \mathcal{F}$ *satisfy* $\mu(E_j \setminus E_{j+1}) = 0$ *for all* $j \geq 1$, *then*

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

Proof

Define $F_n = \bigcap_{j\geq n} E_j$. Then $F_1 \subseteq F_2 \subseteq F_3 \subseteq ...$ and so, by Lemma 4.1,

$$\mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \lim_{n\to\infty} \mu(F_n). \tag{2}$$

Now

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} \bigcap_{j\geq n} E_j \subseteq \bigcup_{n=1}^{\infty} E_n.$$

If $x \in \bigcup_{n=1}^{\infty} E_n$ then there exists $k \geq 1$ such that $x \in E_k$. If $x \notin \bigcup_{n=1}^{\infty} F_n$ then, in particular, $x \notin F_k = \bigcap_{j\geq k} E_j$. So there exists $j \geq k$ such that $x \notin E_j$ (obviously $j \neq k$). Let ℓ be the largest integer in the range $k \leq \ell < j$ for which $x \in E_\ell$. Then $x \notin E_{\ell+1}$ and so $x \in E_\ell \setminus E_{\ell+1}$. Hence

$$\left(\bigcup_{n=1}^{\infty} E_n\right) \setminus \left(\bigcup_{n=1}^{\infty} F_n\right) \subseteq \bigcup_{\ell=1}^{\infty} (E_\ell \setminus E_{\ell+1}).$$

Since the right hand side has measure zero and μ is complete we deduce that

$$\mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right). \tag{3}$$

Obviously $F_n \subseteq E_n$ but what of $E_n \setminus F_n$? Similar to above, if $x \in E_n$ and $x \notin F_n$ then there exists $j \geq n$ such that $x \in E_j$ and so $x \in E_\ell \setminus E_{\ell+1}$ for some $n \leq \ell < j$. That is, $E_n \setminus F_n \subseteq \bigcup_{\ell=1}^{\infty} (E_\ell \setminus E_{\ell+1})$, so $\mu(E_n) = \mu(F_n)$. Combining this with (2) and (3) gives

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

■

Proposition 1

If s is a simple $\mathcal{F}$-measurable function on $\bigcup_{n=1}^{\infty} E_n$, with E_n as in the result above, then

$$\lim_{n\to\infty} I_{E_n}(s) = I_{\bigcup_{n=1}^{\infty} E_n}(s).$$

Proof Straightforward, identical to the proof of Theorem 4.2(v). ■

We can now return to the proof of the Theorem 1. As in the proof of Theorem 4.11

$$\begin{aligned} \int_E f_n d\mu &\geq \int_{E_n} f_n d\mu \\ &\geq \int_{E_n} cs d\mu = cI_{E_n}(s). \end{aligned} \tag{4}$$

We have seen above that the sets E_n satisfy the conditions of Lemma 1 so we let $n \to \infty$ in (4), applying Proposition 1 and obtaining

$$L \geq cI_{\bigcup_{n=1}^{\infty} E_n}(s).$$

What is $\bigcup_{n=1}^{\infty} E_n$?

Consider $x \in E \setminus (\bigcup_{n=1}^{\infty} E_n)$ in which case $cs(x) > f_n(x)$ for all n. If we restrict to $x \in E \setminus (A \cup B)$ then $x \notin A$ which implies that $\lim_{n\to\infty} f_n(x)$ exists, so we have that $cs(x) \geq \lim_{n\to\infty} f_n(x)$. And since $x \notin B$ we have $\lim_{n\to\infty} f_n(x) = f(x)$ and so $cs(x) \geq f(x)$. This is impossible since for all x we have $s(x) \leq f(x)$ and $c < 1$. Hence

$$E \setminus \left(\bigcup_{n=1}^{\infty} E_n \right) \subseteq A \cup B.$$

Since the right hand side has measure zero we conclude that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n \right) = \mu(E)$$

and

$$L \geq cI_E(s).$$

As in the previous version we let $c \to 1$ to get $L \geq I_E(s)$. Thus L is **an** upper bound on the set of integrals of simple functions less than f. Yet $\int_E f d\mu$ is **the** least of all such upper bounds. Hence

$$L \geq \int_E f d\mu. \tag{5}$$

Combining (1) and (5) gives the required equality. ■

Finally, you can never have too many proofs of the following result.

Example

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Verification Let $z = r(\cos x + i \sin x)$ for $0 < r < 1$.

$$\begin{aligned}
\frac{1-r^2}{1-(z+\overline{z})+r^2} &= \frac{1-z\overline{z}}{(1-z)(1-\overline{z})} = \frac{1-\overline{z}+\overline{z}(1-z)}{(1-z)(1-\overline{z})} \\
&= \frac{1}{(1-z)} + \overline{z}\frac{1}{(1-\overline{z})} \\
&= \sum_{n=0}^{\infty} z^n + \overline{z}\sum_{n=0}^{\infty} \overline{z}^n \\
&= 1 + \sum_{n=1}^{\infty} (z^n + \overline{z}^n) \\
&= 1 + 2\sum_{n=1}^{\infty} r^n \cos nx.
\end{aligned}$$

So as in example 20 we can use Lebesgue's Dominated Convergence Theorem to justify

$$\int_a^b f(x)\frac{1-r^2}{1-2r\cos x + r^2}dx = \int_a^b f(x)dx + 2\sum_{n=1}^{\infty} r^n \int_a^b f(x)\cos nx dx,$$

as long as f is finite and integrable over (a, b). Apply this with $f(x) = x^2$ to get

$$\int_0^{\pi} x^2 \frac{1-r^2}{1-2r\cos x + r^2}dx = \frac{\pi^3}{3} + 4\pi\sum_{n=1}^{\infty} \frac{(-1)^n r^n}{n^2},$$

having used integration by parts to evaluate the integrals.

We next try to bound

$$\frac{1+r}{1-2r\cos x + r^2}$$

from above. For $\pi/2 \le x \le \pi$ we have $\cos x \le 0$ and so $1 - 2r\cos x + r^2 \ge 1 + r^2$ in which case

$$\frac{1+r}{1-2r\cos x+r^2} \le \frac{1+r}{1+r^2} \le \frac{\sqrt{2}+1}{2},$$

the maximum value being attained at $r = \sqrt{2} - 1$. For $0 \le x \le \pi/2$ use the inequality

$$\cos x \le 1 - \frac{4}{\pi^2}x^2.$$

(The coefficient $4/\pi^2$ is chosen such that the left hand side equals the right hand side at both $x = 0$ and $x = \pi/2$. I leave it to the student to check that the inequality holds in the interval between but note that when $x = \pi/4$ the left hand side equals $1/\sqrt{2}$ which is less than the value of the right hand side, $3/4$.) Thus

$$\begin{aligned} 1 - 2r\cos x + r^2 &\ge 1 - 2r\left(1 - \frac{4}{\pi^2}x^2\right) + r^2 \\ &= (1-r)^2 + \frac{8rx^2}{\pi^2} \\ &\ge \frac{8rx^2}{\pi^2} \end{aligned}$$

which is a little weak when r is small but we are interested in r near 1. So for $0 \le x \le \pi/2$ we have

$$\frac{1+r}{1-2r\cos x+r^2} \le \frac{\pi^2(r+1)}{8rx^2} \le \frac{\pi^2}{4rx^2}.$$

Then

$$\int_0^\pi x^2 \frac{1-r^2}{1-2r\cos x+r^2}dx = (1-r)\int_0^\pi x^2 \frac{1+r}{1-2r\cos x+r^2}dx$$

$$\begin{aligned} &\le (1-r)\left\{\frac{\pi^2}{4r}\int_0^{\pi/2}\frac{x^2}{x^2}dx + \left(\frac{\sqrt{2}+1}{2}\right)\int_{\pi/2}^{\pi} x^2 dx\right\} \\ &\le (1-r)\left(\frac{1}{8r} + \frac{7(\sqrt{2}+1)}{48}\right)\pi^3 \\ &\le C(1-r) \end{aligned}$$

for some constant $C > 0$ as long as r is not near 0, i.e. $r \geq 1/2$ say. In particular the integral tends to zero as $r \to 1-$. Hence

$$\lim_{r \to 1-} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} r^n}{n^2} = \frac{\pi^2}{12}. \tag{6}$$

We have to be careful here about taking the limit inside the series. Consider

$$\begin{aligned} \sum_{i=0}^{\infty} (-1)^i r^i &= 1 - r + r^2 - r^3 + ... \\ &= \frac{1}{1+r}, \end{aligned}$$

valid for $-1 < r < 1$. So

$$\lim_{r \to 1-} \sum_{i=0}^{\infty} (-1)^i r^i = \lim_{r \to 1-} \frac{1}{1+r} = \frac{1}{2}.$$

Yet if we try take the limit inside the series we get

$$\sum_{i=0}^{\infty} (-1)^i \lim_{r \to 1-} r^i = \sum_{i=0}^{\infty} (-1)^i$$

which is not defined. Of course, the difference with our example is that when the limit is taken inside the series (6) the resulting series is convergent.

Let

$$S(r) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} r^n}{n^2} \quad \text{and} \quad S_k(r) = \sum_{n=1}^{k} \frac{(-1)^{n+1} r^n}{n^2}.$$

By the comparison test $S(r)$ converges (absolutely) for $-1 \leq r \leq 1$.

Consider

$$\begin{aligned} |S(1) - S(r)| &= |S(1) - S_k(1) + S_k(1) - S_k(r) + S_k(r) - S(r)| \\ &\leq |S(1) - S_k(1)| + |S_k(1) - S_k(r)| + |S_k(r) - S(r)|. \end{aligned} \tag{7}$$

Let

$$A(r, M, N) = \sum_{n=M}^{N} \frac{(-1)^{n+1} r^n}{n^2}.$$

Then given any $\varepsilon > 0$ we have that there exists N_0 such that

$$|A(r, M, N)| \leq \sum_{n=M}^{N} \frac{1}{n^2} < \varepsilon$$

for all $N > M > N_0$ and all $-1 \leq r \leq 1$. Fix such an M, and let N tend to ∞. Then with $k = M$ in (7) we see that the first and third terms are less than ε. For the second term we have that $S_M(r)$ is a finite sum of continuous functions and so continuous. Therefore there exists $\delta > 0$ such for $-\delta < |r - 1| < \delta$ we have $|S_M(1) - S_M(r)| < \varepsilon$. Combining we see that there exists $\delta > 0$ such for $1 - \delta < r \leq 1$ we have $|S(1) - S(r)| < \varepsilon$. Hence $\lim_{r \to 1-} S(r) = S(1)$, that is,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Using partial sums it is possible to make the following "suggestion" logically sound.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} &= \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} - \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - 2 \sum_{\substack{n=1 \\ n \text{ even}}}^{\infty} \frac{1}{n^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{2}{2^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}.
\end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

■