Question Sheet 9

1) Let $A = \{a, b, c, d\}$ and let $\mathcal{R}$ be the relation given by

$$\mathcal{R} = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (b, c), (a, c)\}.$$

Draw the digraph of $\mathcal{R}$.

Is $\mathcal{R}$ reflexive, symmetric, transitive? In each case give a counter example if the answer is no.

2) Let $A = \mathbb{Z}$ and let $\mathcal{R} = \{(x, y) : x, y \in A, x^2 = y^2\}$.

Prove that $\mathcal{R}$ is an equivalence relation.

3) Let $A = \mathbb{Z}$ and let $\mathcal{R} = \{(x, y) : xy + y^2 = x^2 + 1\}$.

Which of the following are true?

(i) $0\mathcal{R}0$,

(ii) $1\mathcal{R}1$,

(iii) $0\mathcal{R}1$,

(iv) $1\mathcal{R}0$,

(v) $1\mathcal{R}(-2)$,

(vi) $0\mathcal{R}(-2)$,

(vii) $3\mathcal{R}2$.

Show that $\mathcal{R}$ is not reflexive, not symmetric and not transitive. (Give counterexamples.)

4) Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3\}$. Define a function $f : A \to B$ by the rule

$$f(1) = 1,\ f(2) = 3,\ f(3) = 1 \text{ and } f(4) = 2.$$

What is the image of 3?

What is the codomain of f?

Draw a picture to show f.

Is f one-to-one?

Is f onto?

5) Let $A = \mathbb{N}$ and $B = \mathbb{Q}$. Define a function $f : A \to B$ by the rule

$$f(x) = \frac{3}{2}x \qquad \text{for all } x \in A.$$

Prove that f is one-to-one.

Prove that $f(x) \neq 1$ for all $x \in A$.

Deduce that f is not onto.

6) (i) Define $f : \mathbb{R} \to \mathbb{R}$ by the rule

$$f(x) = x^2 \qquad \text{for all } x \in \mathbb{R}.$$

Show that f is not one-to-one and not onto.

(ii) Let A be the set of all *positive* real numbers. Define $g : A \to A$ by the rules

$$g(x) = x^2 \qquad \text{for all } x \in A.$$

Prove that g is one-to-one.

Is g onto?

7) Define $f : \mathbb{R} \to \mathbb{R}$ by the rule

$$f(x) = 2x - 3 \qquad \text{for all } x \in \mathbb{R}.$$

Prove that f is one-to-one.

Show that

$$f\left(\frac{y+3}{2}\right) = y \qquad \text{for all } y \in \mathbb{R}.$$

Hence prove that f is onto.

8) Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$. Define a function $f : A \to B$ by the rule

$$f\ (1) = b,\ \ f\ (2) = a,\ \ f\ (3) = c,\ \ f\ (4) = a.$$

Let $g : B \to A$ be the function defined by the rule

$$g(a) = 2,\ \ g(b) = 3,\ \ g(c) = 1,\ \ g(d) = 3.$$

Find $(g \circ f)(x)$ for each element x of A.
What is the domain of $g \circ f$?
What is the codomain of $g \circ f$?

9) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be the functions defined by the rules

$$f(x) = \frac{x+1}{2} \quad \text{and} \quad g(x) = \frac{x+1}{2} \quad \text{for all } x \in \mathbb{R}.$$

Find $(g \circ f)(0), (g \circ f)(1)$ and $(g \circ f)(-1)$.
Show that

$$(g \circ f)(x) = \frac{x+3}{4}$$

for all $x \in \mathbb{R}$.

10) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be the functions defined by the rules

$$f(x) = x^2 - 1 \quad \text{and} \quad g(x) = x^2 + 1 \quad \text{for all } x \in \mathbb{R}.$$

Find $(g \circ f)(0)$, $(g \circ f)(1)$, $(f \circ g)(0)$ and $(f \circ g)(1)$.
Find expressions for $(g \circ f)(x)$ and $(f \circ g)(x)$.

11) Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4, 5\}$.
(i) How many functions are there from A to B?
(ii) How many one-to-one functions are there from A to B?
(iii) How many functions from A to B do **not** take a to 1?
(iv) How many one-to-one functions from A to B take a to 1?