2 Natural Deduction

A deductive proof is a step-by-step demonstration that a given argument is valid. At each step we apply *rules of inference.* We will justify the introduction of these rules by using truth tables to show that they arise from valid arguments, but it should be stressed that a proof of validity by deduction has no connection to a proof of validity by truth tables. We will come back to this point in Section 2.4.

The following rules should be memorised.

2.1 Rules I

A Rule of Assumption

At any step of the proof we can introduce any premise.

Note: Premises may be used more that once, or need not be used at all.

M.P.P. (Modus Ponendo Ponens) (Translate as: the method of affirming.)

$p,\ p \rightarrow q \vdash q$ is valid, as can be seen by using truth tables (Ex 19(i)), so

If steps of the form p and $p \rightarrow q$ occur in the proof, then we can deduce q.

Note: we use the word 'form' because p and q might be built up from smaller propositions.

M.T.T. (Modus Tollendo Tollens) (Translate as: the method of denying.)

$\neg\, q,\ p \rightarrow q \vdash \neg\, p$ is valid, (see Ex 19(ii)), so

If steps of the form $\neg\, q$ and $p \rightarrow q$ occur in the proof, then we can deduce $\neg\, p$.

D.N. (Double Negative)

$p \equiv \neg\, (\neg\, p))$, so

If steps of the form $\neg\, (\neg\, p)$ occurs in the proof, then we can deduce p, and vice-versa.

Example 22 (i) Show that the following argument is valid.

$(A \vee B) \rightarrow (C \vee D),\ A \vee B,\ (C \vee D) \rightarrow G \ \vdash\ G.$

1	$A \vee B$	A
2	$(A \vee B) \rightarrow (C \vee D)$	A
3	$C \vee D$	MPP 1,2
4	$(C \vee D) \rightarrow G$	A
5	G	MPP 3,4

Therefore the argument is valid.

(ii) Show that the following argument is valid.

$(A \vee B) \rightarrow (C \vee D), \neg\,(C \vee D), (\neg\, G) \rightarrow (A \vee B) \ \vdash\ G.$

1	$\neg\,(C \vee D)$	A
2	$(A \vee B) \rightarrow (C \vee D)$	A
3	$\neg\,(A \vee B)$	MTT 1,2
4	$(\neg\, G) \rightarrow (A \vee B)$	A
5	$\neg\ (\neg\, G)$	MTT 3,4
6	G	DN 5

Example 21 (again) Show that the following argument is valid.

$p \rightarrow (s \rightarrow (\neg\, r)),\ p \rightarrow r,\ p \ \vdash\ \neg\, s$

1	p	A
2	$p \rightarrow r$	A
3	r	MPP 1,2
4	$p \rightarrow (s \rightarrow (\neg\, r))$	A
5	$s \rightarrow (\neg\, r)$	MPP 1,4
6	$\neg\,(\neg\, r)$	DN 3
7	$\neg\, s$	MTT 5,6

Therefore the argument is valid.

(Note how quick the proof is compared to using a truth table.)