1.4 Arguments

Definition

A *valid argument* is a finite set of propositions $P_1, ..., P_r$ called *premises*, together with a proposition C, the *conclusion*, such that the propositional form $(P_1 \wedge P_2 \wedge ... \wedge P_r) \rightarrow C$ is a tautology.

We say C *follows logically from,* or is a *logical consequence of* the premises.

We write $P_1, ..., P_r \ \vdash \ C$. The symbol $\vdash$ is called the *turnstile*.

Definition

If an argument in **not** valid we say that it is *invalid*.

First Method to prove validity

Example 19

Let $Q_1 =$ "John graduates"

$Q_2 =$ "Mary graduates"

$Q_3 =$ "John gets a job"

$Q_4 =$ "Mary gets a job"

$Q_5 =$ "Mary earns money"

(i) Consider the following argument:

"If John graduates then he gets a job".
"John graduates".
"Therefore John gets a job".

To see the "form" of this argument we symbolize it as $Q_1 \rightarrow Q_3, \ \ Q_1 \ \ \vdash Q_3$.

Now we check that $((Q_1 \rightarrow Q_3) \wedge Q_1) \rightarrow Q_3$ is a tautology:

Q_1	Q_3	$Q_1 \rightarrow Q_3$	$(Q_1 \rightarrow Q_3) \wedge Q_1$	$((Q_1 \rightarrow Q_3) \wedge Q_1) \rightarrow Q_3$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

This tautology shows that the argument is valid.

Note Another "instance" of this argument follows if we set Q_1 = "$2 < 1$" and Q_3 = "$3 < 2$". The argument then reads:

"If $2 < 1$ then $3 < 2$"
"$2 < 1$"
"Therefore, $3 < 2$".

This is still valid though some of the propositions, i.e. $2 < 1$ for instance, are false.

Example 19 continued. (ii) Consider the following argument:

"If Mary graduates then she gets a job".
"Mary does not get a job".
"Therefore Mary does not graduate".

Symbolized, this becomes $Q_2 \to Q_4, \ (\neg\, Q_4) \ \vdash \ (\neg\, Q_2)$.

*Now check that $((Q_2 \to Q_4) \wedge (\neg\, Q_4)) \to (\neg\, Q_2)$ is a tautology.

Q_2	Q_4	$Q_2 \to Q_4$	$\neg\, Q_4$	$(Q_2 \to Q_4) \wedge (\neg\, Q_4)(\equiv A)$	$\neg\, Q_2$	$A \to (\neg\, Q_2)$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

(iii) Consider the following argument:

"Either Mary or John graduate".
"John does not graduate".
"Therefore Mary graduates".

Symbolized, this becomes $Q_2 \vee Q_1, \ (\neg\, Q_1) \ \vdash \ Q_2$.

*Now check that $((Q_2 \vee Q_1) \wedge (\neg\, Q_1)) \to Q_2$ is a tautology.

Q_1	Q_2	$Q_2 \vee Q_1$	$\neg\, Q_1$	$(Q_2 \vee Q_1) \wedge (\neg\, Q_1)(\equiv B)$	$B \to Q_2$
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

(iv) Consider the following argument:

"If Mary graduates then she gets a job".
"If Mary gets a job then she earns money".
"Therefore if Mary graduates then she earns money".

Symbolized, this becomes $Q_2 \to Q_4,\ Q_4 \to Q_5\ \vdash\ (Q_2 \to Q_5)$.

*Now check that $((Q_2 \to Q_4) \wedge (Q_4 \to Q_5)) \to (Q_2 \to Q_5)$ is a tautology.

Q_2	Q_4	Q_5	$Q_2 \to Q_4$ A	$Q_4 \to Q_5$ B	$A \wedge B$	$Q_2 \to Q_5$ C	$(A \wedge B) \to C$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

We can sum up the above by saying the following are all valid:

(i) $p \to q,\ p\ \vdash\ q$,

(ii) $p \to q,\ \neg\, q\ \vdash\ \neg\, p$,

(iii) $p \vee q,\ \neg\, q\ \vdash\ p$,

(iv) $p \to q,\ q \to r\ \vdash\ p \to r$.

Example Show that $p \to q,\ q \vee p\ \vdash\ (\neg\, q) \vee (\neg\, p)$ is valid.

p	q	$p \to q$	$q \vee p$	$((p \to q) \wedge (q \vee p))$ P	$((\neg\, q) \vee (\neg\, p))$ C	$P \to C$	
T	T	T	T	T	F	F	←
T	F	F	T	F	T	T	
F	T	T	T	T	T	T	
F	F	T	F	F	T	T	

We do not have a tautology in the last column so the argument is invalid.

Second Method to prove validity

Note If an argument is valid then $(P_1 \wedge P_2 \wedge ... \wedge P_r) \to C$ is a tautology, and so it is always true. So we need to prove that it is never false. It can only be false if C is false and $P_1 \wedge P_2 \wedge ... \wedge P_r$ is true, i.e. all P_1, P_2,...,P_r are true. So we **never** want to see a row in the truth table where all the premises are true and the conclusion false.

This observation gives a **second** way of checking that an argument is valid or not.

Example 20

Is $p \rightarrow q,\ q \vee p \vdash (\neg\, q) \vee (\neg\, p)$ valid?

We look at the truth table:

p	q	$p \rightarrow q$	$q \vee p$	$(\neg\, q) \vee (\neg\, p)$	
T	T	T	T	F	←
T	F	F	T	T	
F	T	T	T	T	
F	F	T	F	T	

In the first line the conclusion is false, but all premises are true. Hence the argument is *invalid.*

This method requires fewer columns than in the first method.

Is there an "instance" of this argument which is "obviously" invalid?

Try looking in the "World of Mathematics", for instance, choosing $p \equiv$ "$3 > 2$" and $q \equiv$ "$2 > 1$". Then the argument becomes:

If $3 > 2$ then $2 > 1$,
Either $3 > 2$ or $2 > 1$,
Therefore, either $3 \leq 2$ or $2 \leq 1$.

Both premises are true but the conclusion is false. On the basis that we never want a false conclusion to follow from true premises, this argument is invalid. What we have here is an example where p is True and q is True, which is the first line in the table, where we had seen the problem.

But be careful! Consider another instance. So let $p \equiv$ "Stockport is a city" and $q \equiv$ "Manchester is a city". Then the argument becomes:

If Stockport is a city then Manchester is a city,
Either Manchester or Stockport is a city,
Therefore, either Manchester is not a city or Stockport is not a city.

If I tell you that Manchester is a city but Stockport is not a city then you can check that all the propositions in this argument are true. But the argument is still invalid. It is a case of the conclusion, though true, not following logically from the true premises.

Example 21 Is $p \to (s \to (\neg\, r)),\ p \to r,\ p\ \vdash\ \neg\, s$ valid?

p	r	s	$\neg\, r$	$(s \to (\neg\, r)$	$p \to (s \to (\neg\, r))$	$p \to r$	p	$\neg\, s$
T	T	T	F	F	F	T	T	F
T	T	F	F	T	T	T	T	T
T	F	T	T	T	T	F	T	F
T	F	F	T	T	T	F	T	T
F	T	T	F	F	T	T	F	F
F	T	F	F	T	T	T	F	T
F	F	T	T	T	T	T	F	F
F	F	F	T	T	T	T	F	T

We look at each row in turn.

We look to see if on any row we have a case of all the premises being true with the conclusion false.

For instance in the first row we see that premises are F, T, T and the conclusion F. This is allowable. By checking each row we see that each row is allowable, that is, we never have a case of all premises true with the conclusion false.

Hence the argument is valid.

This method is straightforward. In fact, a machine can do it.

Note I have given here **two** methods for using a truth table to check whether $P_1, ..., P_r \vdash C$ is valid or not.

Do not mix up these methods!

In the *first method* use a truth table to work out the truth values of $(P_1 \wedge P_2 \wedge ... \wedge P_r) \to C$, and hope that it is always true, i.e. a tautology.

In the *second method* construct a table containing a column for each of the $P_1, P_2, ...$ up to P_r along with C and hope that there is no row with all the P_i true and C false.

If an argument is **invalid** there is *sometimes* a quick method of showing this.

Example Show that

$$(p \vee q) \to s,\ q \to s\ \vdash\ s$$

is invalid.

We do this by trying to make the conclusion false and the premises all true.

The conclusion if false if we choose s to be false. Then $q \to s$ can be true only if q is false. Finally, for $(p \vee q) \to s$ to be true we require $p \vee q$ to be false, and so p must be false.

Hence if all of p, q and s are false (i.e. the bottom row of the truth table) we see that all the premises are true but the conclusion is false. Hence the argument is invalid.

The second method of proving validity needs a smaller number of columns than the first, but if the number of basic propositions p, q, r, etc. is large then the tables in both methods need a large number of rows. Thus the tables get cumbersome in both methods and an alternative method is necessary.