# 7 Matrices

## 7.1 Definition

**Definition** A *matrix* is a rectangular array of numbers.

**Example 81**

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 20 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

These are examples of $2 \times 3$, $3 \times 1$ and $2 \times 2$ matrices respectively, where the first number is the number of rows and the second the number of columns.

**Note** The number of rows = length of the columns.

The number of columns = lenth of the rows.

We often denote an $m \times n$ matrix $A$ by $(a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$, where $a_{ij}$ is the element on the $i^{th}$ row and $j^{th}$ column. So

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & & & & & & \vdots \\ a_{31} & & & & \vdots & & \vdots \\ \vdots & & & & & & \vdots \\ a_{i1} & & \cdots & & a_{ij} & & \vdots \\ \vdots & & & & & & \vdots \\ a_{m1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{mn} \end{pmatrix}.$$

**Definition** We say that two matrices are *equal*, and write $A = B$ if, and only if, $a_{ij} = b_{ij}$ for all $i, j$. In particular, if the matrices are equal then they must be the same size.

**Definition** If $A$ and $B$ are the same size, $m \times n$ say, then $A + B$ is the $m \times n$ matrix $C = (c_{ij})$ where $c_{ij} = a_{ij} + b_{ij}$ for all $i, j$.

**Example 82**

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 20 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 4 & 0 \\ 10 & 0 & 20 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 3 \\ 11 & 20 & 20 \end{pmatrix}.$$

**Note**

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 20 & 0 \end{pmatrix} + \begin{pmatrix} 10 & -1 \\ 0 & 4 \\ 20 & 0 \end{pmatrix}$$

is not defined since the matrices are different sizes.

**Definition** Let $A$ and $B$ be two matrices such that the length of the rows of $A$ is equal to the length of the columns of $B$, that is $A$ is $r \times s$ and $B$ is $s \times t$ for some $r, s$ and $t$. Then the *scalar product* of the $i^{th}$ row of $A$, $(a_{i1}, a_{i2}, \ldots, a_{is})$ with the $j^{th}$ column of $B$, $(b_{1j}, b_{2j}, \ldots, b_{sj})$, is

$$(a_{i1}, a_{i2}, \ldots, a_{is}) \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{sj} \end{pmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{is}b_{sj} = \sum_{k=1}^{s} a_{ik}b_{kj}.$$

(* Often call $(a_{i1}, a_{i2}, \ldots, a_{is})$ a row vector and

$$\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{sj} \end{pmatrix}$$

a column vector in which case the scalar product is known as a *vector (inner) product.*)

The *matrix product* $AB$ is the $r \times t$ matrix $C = (c_{ij})$ where $c_{ij} = \sum_{k=1}^{s} a_{ik}b_{kj}$ is the scalar product of the $i^{th}$ row of $A$ with the $j^{th}$ column of $B$.

**Example 83 (i)** Scalar product.

$$(1, 2, -3) \begin{pmatrix} 5 \\ 6 \\ 2 \end{pmatrix} = 1 \times 5 + 2 \times 6 + (-3) \times 2 = 5 + 12 - 5 = 11.$$

Yet

$$(1, 2) \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

is not defined since the row and columns are of different lengths.

**(ii)** Let

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 8 & 5 & 3 \\ -5 & 1 & 0 \end{pmatrix}, \text{ then}$$

$$\begin{aligned} AB &= \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 8 & 5 & 3 \\ -5 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 8 \\ -5 \end{pmatrix} & \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} & \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 5 \end{pmatrix} \begin{pmatrix} 8 \\ -5 \end{pmatrix} & \begin{pmatrix} 1 & 5 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 2 \times 8 + 3 \times (-5) & 2 \times 5 + 3 \times 1 & 2 \times 3 + 3 \times 0 \\ 1 \times 8 + 5 \times (-5) & 1 \times 5 + 5 \times 1 & 1 \times 3 + 5 \times 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 13 & 6 \\ -17 & 10 & 3 \end{pmatrix}. \end{aligned}$$

**Note** $BA$ is not defined since the rows of $B$ are a different length to the columns of $A$.

**Example 84** Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix},$$

and

$$BA = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}.$$

So $AB$ and $BA$ are different sizes and so cannot be equal. Thus matrix multiplication is **not** commutative.

It can be shown that matrix multiplication is associative, so **if** $A$, $B$ and $C$ are matrices for which $(AB)C$ is defined **then** $A(BC)$ is defined **and**

$$(AB)C = A(BC).$$

We also have the distributive property so that if $A$ is $m \times q$ and $B, C$ are both $q \times n$, for some $m, n$ and $q$, then

$$A(B + C) = AB + AC.$$

## 7.2 Identity

In $\mathbb{R}$ the number 1 has a special property, namely that $1x = x$ for all $x \in \mathbb{R}$. We say that 1 is a *multiplicative identity*, because it leaves unchanged any number multiplied by it.

(*Question for students: what is the additive identity in $\mathbb{R}$?)

For matrices we note that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 1 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 4 & 5 \end{pmatrix}$
and

$$\begin{pmatrix} 4 & 5 & 6 \\ -1 & 0 & 2 \\ -4 & -3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \\ -1 & 0 & 2 \\ -4 & -3 & 2 \end{pmatrix}.$$

**Definition**

The matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, etc., are *Identity matrices.*

Such matrices are always square with 1 on the *leading diagonal* and 0 elsewhere.

The $m \times m$ identity is denoted by $I_m$.

For a general $m \times r$ matrix $A$ we will always have $I_m A = A$ while for any $s \times m$ matrix $B$ we will have $BI_m = B$.

## 7.3 Inverses

For any non-zero number $x \in \mathbb{R}$ its *inverse* is that number $y$ such that if you multiply $x$ by $y$ you get the identity. That is $xy = 1$. For example, the inverse of 3 is 1/3.

The same idea holds for matrices, but given a matrix $A$, if it has an inverse $B$ then, because matrix multiplication is not commutative, we need to check that both $AB = I$ and $BA = I$. We restrict to square matrices. So

**Definition** A square $n \times n$ matrix $A$ has a *multiplicative inverse*, denoted by $A^{-1}$, if

$$AA^{-1} = I_n \text{ and } A^{-1}A = 1_n$$

We will see later that not all square matices have an inverse.

**Example 85** The inverse of

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 1 & 3 & 2 \end{pmatrix} \quad \text{is} \quad \begin{pmatrix} 8 & 5 & -7 \\ -2 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix}.$$

**Question** How do we know this?

**Answer** We "verify" the definition. That is, we check that

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 8 & 5 & -7 \\ -2 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix} = I_3 \quad \text{and} \quad \begin{pmatrix} 8 & 5 & -7 \\ -2 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 1 & 3 & 2 \end{pmatrix} = I_3.$$

I leave it to the students to check this.

**Question** Why are inverses useful?

**One of many possible Anwers:**

## 7.4 Solving systems of linear equations

**Example 86** Let $U = \mathbb{R}$. Then

$$\begin{aligned} &6x &&= 1, \\ &5x + 7y &&= 3, \\ &\tfrac{1}{2}x_1 + 33.2x_2 + 15x_3 &&= \frac{33}{4}, \end{aligned}$$

are all *linear* equations because the variables are not multiplied together and are not raised to any power different from 1.

**Example 86**

$$\begin{aligned} xy &= 1, \\ x^2 + y^2 &= 2, \end{aligned}$$

are **not** linear equations.

### 7.4.1 One Equation in one unknown

**Example 88** Consider $6x = 2$.

Multiply both sides by the inverse of 6, i.e. $6^{-1}$ or $\frac{1}{6}$ to get

$$\tfrac{1}{6}(6x) = \tfrac{1}{6}2,$$

$$\text{that is} \quad x = \tfrac{1}{3}.$$

## 7.4.2 Three Equations in three unknowns

**Example 89** Find three real numbers $x$, $y$ and $z$ that simultaneously satisfy

$$\begin{aligned} x + 2y + 3z &= 60 \\ y - 2z &= 0 \\ x + 3y + 2z &= -4 \end{aligned}$$

**Solution** The "trick" here is to write the system as one matrix equation.

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 60 \\ 0 \\ -4 \end{pmatrix}.$$

This eqation could be written as $A\mathbf{x} = \mathbf{c}$ where $A, \mathbf{x}$ and $\mathbf{c}$ are a matrices (* $\mathbf{x}, \mathbf{c}$ are also called *column vectors*).

Then just as in 7.4.1, if the inverse, $A^{-1}$ exists, we can multiply both sides of the equation to get

$$A^{-1}\mathbf{c} = A^{-1}(A\mathbf{x}) = \left(A^{-1}A\right)\mathbf{x} = I\mathbf{x} = \mathbf{x}.$$

So the solution is given by $\mathbf{x} = A^{-1}\mathbf{c}$.
In this example

$$A^{-1}\mathbf{c} = \begin{pmatrix} 8 & 5 & -7 \\ -2 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 60 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 508 \\ -128 \\ -64 \end{pmatrix}.$$

So the solution is $x = 508, y = -128$ and $z = -64$.

**Question** How do we know this is correct?

**Answer** Substitute it back in the original system. You should always, always do this!

I leave it to the student to do this.

**Question** How do we find inverses?

**One of many possible Anwers:**

## 7.5 Gaussian Elimination.

I will describe this method by way of an example.

**Example 90** Find the inverse of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 1 & 3 & 2 \end{pmatrix}.$$

**Solution**. Start with the *augmented matrix*

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right) = (A|\, I_3)\,.$$

Our aim is to use *row operations,* consisting of

1. Multiplying a row by a non-zero scalar,
2. Replacing a row by the sum of it and another row,
3. Exchanging rows,

to tranform $A$ into $I_3$. When these row operations are applied to the augmented matrix they will transform the $I_3$ part. In fact it will be transformed into the inverse. (*No proof of this is give. You need to take it on trust.).

Lets see this in action.

$$r_3 \to r_3 - r_1 \qquad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$r_3 \to r_3 - r_2 \qquad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

$$\begin{array}{l} r_1 \to r_1 - 3r_3 \\ r_2 \to r_2 + 2r_3 \end{array} \qquad \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & 3 & -3 \\ 0 & 1 & 0 & -2 & -1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

$$r_1 \to r_1 - 2r_2 \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 5 & -7 \\ 0 & 1 & 0 & -2 & -1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right)$$

Hence

$$A^{-1} = \begin{pmatrix} 8 & 5 & -7 \\ -2 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix}.$$

**Question** How do I know in what order to apply the operations?

**Answer** There is no unique answer but you always keeps some aims in mind.

So, given a matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

you apply operations to make, *in this order*, a 1 in the 1-1 position (called the *pivot*). Then 0's below it, in the 2-1, 3-1 and 4-1 position. Then go to the next pivot and get a 1 in the 2-2 position. Next get 0's below in the 3-2 and 4-2 postion. Onto the next pivot and get a 1 in the 3-3 position and 0's below it at the 4-3 position. Next a 1 at the last pivot, the 4-4 position. So half way through the matrix looks like

$$\begin{pmatrix} 1 & b_{12} & b_{13} & b_{14} \\ 0 & 1 & b_{23} & b_{24} \\ 0 & 0 & 1 & b_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Finish off by making 0's above the last pivot, i.e. 0's in the 3-4, 2-4 and 1-4 position. Then onto the third column, i.e. 0's in the 2-3 and 1-3 position. Finally a 0 in the 1-2 position.

Always, always approach the problem in this order.

**Example 91** Find the inverse of

$$\begin{pmatrix} 8 & 4 & 1 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix}$$

**Solution** Consider

$$\left(\begin{array}{ccc|ccc} 8 & 4 & 1 & 1 & 0 & 0 \\ 4 & 3 & 2 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

We need to put a 1 in the 1-1 position. We could multiply row 1 by 1/8, but this would lead to fractions and I suggest you try to avoid fractions. They increase the possibilty of arithmetic errors. Instead, note the $4-3=1$. So try

$$r_2 \to r_2 - r_3 \qquad \left(\begin{array}{ccc|ccc} 8 & 4 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & -1 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right),$$

$$r_1 \leftrightarrow r_2 \qquad \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & -1 \\ 8 & 4 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right).$$

Now have to put 0's below the first pivot. So

$$r_2 \to r_2 - 8r_1 \qquad \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & -1 \\ 0 & -4 & -7 & 1 & -8 & 8 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$r_3 \to r_3 - 3r_1 \qquad \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & -1 \\ 0 & -4 & -7 & 1 & -8 & 8 \\ 0 & -1 & -2 & 0 & -3 & 4 \end{array}\right)$$

Now a 1 in the next pivot, the 2-2 position. Try

$$r_2 \to r_2 - 5r_3 \qquad \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & 1 & 7 & -12 \\ 0 & -1 & -2 & 0 & -3 & 4 \end{array}\right)$$

**Note** In the last step we had the calculation $-4 - 5(-1)$. Be careful about these double negatives. Students often make arithmetic errors because of them.

Now a zero below the 2-2 pivot, i.e. a 0 in the 3-2 position.

$$r_3 \to r_3 + r_2 \qquad \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & 1 & 7 & -12 \\ 0 & 0 & 1 & 1 & 4 & -8 \end{array}\right)$$

We already have a 1 in the 3-3 pivot.
So need only get 0's above it in the 2-3 and 1-3 positions.
Why not do both at once?

$$\begin{array}{l} r_2 \to r_2 - 3r_3 \\ r_1 \to r_1 - r_3 \end{array} \qquad \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & -3 & 7 \\ 0 & 1 & 0 & -2 & -5 & 12 \\ 0 & 0 & 1 & 1 & 4 & -8 \end{array}\right)$$

We already have a 1 in the 2-2 pivot.
So need only get a 0 above it.

$$r_1 \to r_1 - r_2 \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -5 \\ 0 & 1 & 0 & -2 & -5 & 12 \\ 0 & 0 & 1 & 1 & 4 & -8 \end{array}\right).$$

Thus the inverse is

$$\begin{pmatrix} 1 & 2 & -5 \\ -2 & -5 & 12 \\ 1 & 4 & -8 \end{pmatrix}.$$

**Remember** Always, always check your anwer by multiplying out.

**Example 92** Solve the system of equations

$$\begin{aligned} 8x + 4y + z &= 2 \\ 4x + 3y + 2z &= 3 \\ 3x + 2y + z &= -2. \end{aligned}$$

**Solution** This system can be written as

$$\begin{pmatrix} 8 & 4 & 1 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$$

So the answer is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & -5 \\ -2 & -5 & 12 \\ 1 & 4 & -8 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 18 \\ -43 \\ 30 \end{pmatrix}$$

**Remember**. Always, always check your answer by substituting back in.

**Example 93** Find the inverse of

$$\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$$

**Solution** Consider

$$\left( \begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -2 & -4 & 0 & 1 \end{array} \right).$$

We have a 1 in the 1-1 pivot so we next get a 0 in the position below it. This can be done in only one way.

$r_2 \to r_2 + 2r_1$ $$\left( \begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array} \right).$$

This row of zeros shows that we will never be able to get the identity in the first half of the augmented matrix. This is an example of a matrix with no inverse.

## *Additional Material (Not for examination).

If we simply want to solve a system of equations and not find the inverse of the matrix we can do the following:

**Example 94** Find real numbers $x$, $y$ and $z$ that simultaneously satisfy

$$\begin{aligned} 2x - 8y + 37z &= 101 \\ x - 3y + 15z &= 41 \\ -x + 4y - 17z &= -46. \end{aligned}$$

**Solution**. Consider a different type of augmented matrix:

$$\left(\begin{array}{rrr|r} 2 & -8 & 37 & 101 \\ 1 & -3 & 15 & 41 \\ -1 & 4 & -17 & -46 \end{array}\right)$$

First aim: To start from the **left** and first get a 1 in 1-1 position.

We can do this in at least three different ways.

(1) Multiplying $r_1$ by $\frac{1}{2}$,

(2) adding the third row to the first, or

(3) exchange the first and second rows.

I do not like (1) since I do not like to have fractions unless I have to have them. I do not like (2) since the more operations we do, such as addition, the more chance there is for a mistake. This leaves us with (3).

$$\begin{array}{c} r_1 \longleftrightarrow r_2 \\ \text{swap} \end{array} \quad \left(\begin{array}{rrr|r} 1 & -3 & 15 & 41 \\ 2 & -8 & 37 & 101 \\ -1 & 4 & -17 & -46 \end{array}\right)$$

Next aim: get 0's below the 1-1 pivot by applying row operations to $r_2$ and $r_3$.

$$\begin{array}{l} r_2 \to r_2 - 2r_1 \\ r_3 \to r_3 + r_1 \end{array} \qquad \left(\begin{array}{rrr|r} 1 & -3 & 15 & 41 \\ 0 & -2 & 7 & 19 \\ 0 & 1 & -2 & -5 \end{array}\right)$$

Aim: we now look in the second column and try to get a 1 in the 2-2 position. This we do by swapping the second and third rows.

$$r_2 \longleftrightarrow r_3 \qquad \left(\begin{array}{rrr|r} 1 & -3 & 15 & 41 \\ 0 & 1 & -2 & -5 \\ 0 & -2 & 7 & 19 \end{array}\right)$$

Next we aim to get zeros in that part of the column below the diagonal entry. There is only one position in that part of the column, namely the 3-2 position. We make this position zero by adding twice the second row to the third row.

$$r_3 \longleftrightarrow r_3 + 2r_2 \qquad \left(\begin{array}{rrr|r} 1 & -3 & 15 & 41 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 3 & 9 \end{array}\right) \tag{1}$$

We now move to the third column and first try to get a 1 in the diagonal position, i.e. the 3-3 position. We already have a non-zero term on the diagonal in the third column. We can make this value 1 by multiplying $r_3$ by a third. I don't mind multiplying by a fraction because the only other non-zero term is the 9 which becomes 3, an integer.

$$r_3 \to \tfrac{1}{3} r_3 \qquad \left(\begin{array}{ccc|c} 1 & -3 & 15 & 41 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Next aim: get 0's above the final pivot.

$$\begin{array}{l} r_1 \to r_1 - 15r_3 \\ r_2 \to r_2 + 2r_3 \end{array} \qquad \left(\begin{array}{ccc|c} 1 & -3 & 0 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Finally, get 0 in column above the 2-2 pivot

$$r_1 \to r_1 + 3r_2 \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Hence solution is $x = -1, y = 1$ and $z = 3$.

**Always, always** check your answer by substituting back into the original system of equations.

## * Back Substitution

We can speed up the process of solving systems of equations further by stopping when $A$ is reduced to an upper triangular matrix, as we saw in equation (1) above. For if we write out the equations represented by the matrix equation (1), we find

$$\begin{aligned} x - 3y + 15z &= 41 \\ y - 2z &= -5 \\ 3z &= 9. \end{aligned}$$

From the third equation we see that $z = 3$. We can substitute this into the second equation to find that $y - 2 \times 3 = -5$, so $y = 1$.

Substituting these values into the first equation gives $x - 3 \times 1 + 15 \times 3 = 41$, and so $x = -1$.

## * No Inverse

Consider

$$\begin{aligned} x + 2y &= 1 \\ -2x - 4y &= 1. \end{aligned}$$

We do Gaussian elimination on the augmented matrix.

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ -2 & -4 & 1 \end{array}\right), \qquad \text{after} \quad r_2 \to r_2 + 2r_1 \quad \text{we get} \quad \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 3 \end{array}\right).$$

It is impossible to find solutions of the two equations represented by this new augmented matrix. The last line represents the equation $0x + 0y = 3$ which has no solutions. Hence our original system of equations has *no* solutions.

Or, consider

$$\begin{aligned} x + 2y &= 1 \\ -2x - 4y &= -2 \end{aligned}$$

When we do Gaussian elimination this time on

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ -2 & -4 & -2 \end{array}\right), \qquad \text{after} \quad r_2 \to r_2 + 2r_1 \quad \text{we get} \quad \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right).$$

Of the two equations represented here the second reads $0x + 0y = 0$ which is satisfied for all $x$ and $y$. So we are left with just one non-trivial equation, $x + 2y = 1$, which has infinitely many solutions i.e. $(x, y) = (1, 0), (-1, 1), (-3, 2)$ etc. Hence our original system of equations has *infinitely* many solutions.