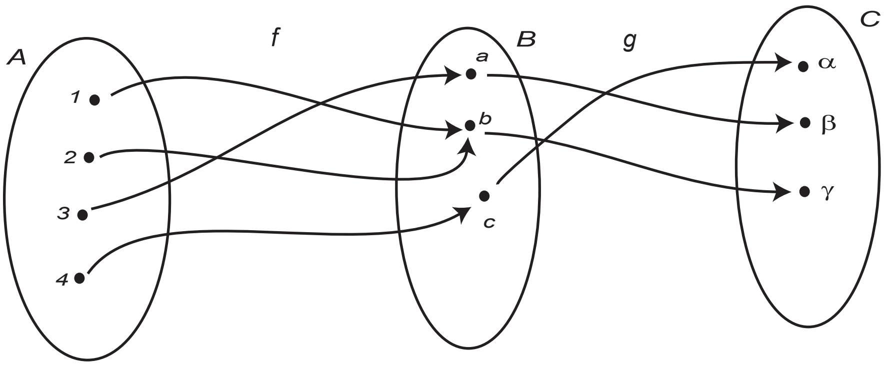
6.4 Composition and Inverses

Definition
If $f\ : A \to B$ and $g : B \to C$ are functions the we define the *composite function*, $g \circ f : A \to C$ by $(g \circ f)(a) = g(f(a))$ for each $a \in A$.

Example 78
(1)



So

$$\begin{aligned} g \circ f\,(1) &= g\,(f\,(1)) = g\,(b) = \gamma, \\ g \circ f\,(2) &= g\,(f\,(2)) = g\,(b) = \gamma, \\ g \circ f\,(3) &= g\,(f\,(3)) = g\,(a) = \beta, \\ g \circ f\,(4) &= g\,(f\,(4)) = g\,(c) = \alpha. \end{aligned}$$

(2) Define $f\ : \mathbb{R} \to \mathbb{R},\ x \mapsto x^2 - 1$ and $g : \mathbb{R} \to \mathbb{R},\ x \mapsto x + 1$. Then

$$(g \circ f\ )(x) = g(f(x)) = g(x^2 - 1) = (x^2 - 1) + 1 = x^2$$

and

$$(f\ \circ g)(x) = f(g(x)) = f(x + 1) = (x + 1)^2 - 1 = x^2 + 2x.$$

Note that $g \circ f(1) = 1^2 = 1$ while $f\ \circ g(1) = 1^2 + 2 = 3$ and so $g \circ f(1) \neq f\ \circ g(1)$ and thus $g \circ f \neq f\ \circ g$. Hence composition of functions need *not* be commutative.

Definition
The *identity function*, 1_A, on a set A satisfies $1_A(a) = a$ for all $a \in A$.

Definition
A function $f : A \to B$ has an *inverse function* if, and only if, there exists a function $g : B \to A$ such that both $g \circ f$, which maps from $A \to B \to A$, i.e. from $A \to A$, is the identity function on A and $f \circ g$, which maps from $B \to A \to B$, i.e. from $B \to B$, is the identity function on B. That is $g \circ f = 1_A$ and $f \circ g = 1_B$. If the inverse function exists we label it as f^{-1}.

Result: If f has an inverse then f is a bijection.

***Proof** (not for examination)

Assume $f : A \to B$ has an inverse.

(i) Given $b \in B$, take $a = f^{-1}(b)$ then

$$\begin{aligned} f(a) &= f(f^{-1}(b)) \\ &= (f \circ f^{-1})(b) \\ &= 1_B(b) \\ &= b. \end{aligned}$$

Thus f maps onto b. This is true for all $b \in B$, so f maps onto B, i.e. f is an onto function.

(ii) By definition of f^{-1} we have

$$(f^{-1} \circ f)(a) = 1_A(a) = a \ \forall a \in A.$$

Assume

$$f(a_1) = f(a_2).$$

Apply f^{-1} to both sides. Then

$$\begin{aligned} & f^{-1}(f(a_1)) = f^{-1}(f(a_2)) \\ \text{i.e} \quad & (f^{-1} \circ f)(a_1) = (f^{-1} \circ f)(a_2) \\ \text{i.e.} \quad & 1_A(a_1) = 1_A(a_2) \\ \text{hence} \quad & a_1 = a_2. \end{aligned}$$

Thus

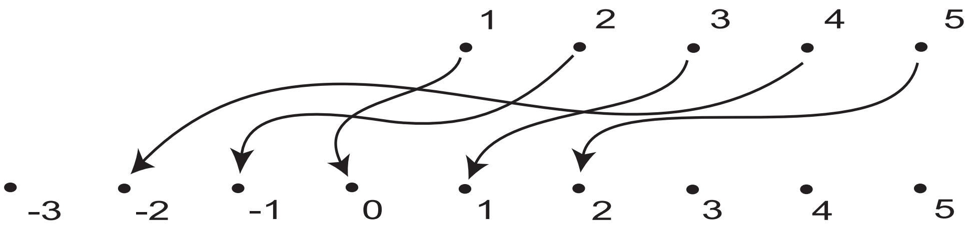
$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2.$$

Therefore f is 1–1.

So, if $f : A \to B$ has an inverse we can conclude that f is a bijection. In other words, only bijections have inverses.

Example 79

Define $f : \mathbb{N} \to \mathbb{Z}$ by the diagram



or, as a formula,

$$f(n) = \begin{cases} \dfrac{n-1}{2} & \text{if } n \text{ is odd} \\ -\dfrac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

This is a bijection and has inverse $g : \mathbb{Z} \to \mathbb{N}$ given by

$$g(n) = \begin{cases} 2n+1 & \text{if } n \geq 0 \\ -2n & \text{if} n < 0. \end{cases}$$

Check that $g \circ f = 1_{\mathbb{N}}$. This requires two cases. In the first n is odd when

$$\begin{aligned} g \circ f(n) &= g(f(n)) \\ &= g\left(\frac{n-1}{2}\right) \\ &= 2\left(\frac{n-1}{2}\right) + 1 && \text{since } \frac{n-1}{2} \geq 0 \\ &= n. \end{aligned}$$

In the second case n is even when

$$\begin{aligned} g \circ f(n) &= g(f(n)) \\ &= g\left(-\frac{n}{2}\right) \\ &= -2\left(-\frac{n}{2}\right) && \text{since } -\frac{n}{2} < 0, \\ &= n. \end{aligned}$$

In all cases $g \circ f(n) = n$ and so $g \circ f = 1_{\mathbb{N}}$.

To check that $f \circ g = 1_{\mathbb{Z}}$ we again have two cases though this time they are $n \geq 0$ and $n < 0$. In the first case

$$\begin{aligned} f \circ g(n) &= f(g(n)) \\ &= f(2n+1), && \text{since } n \geq 0, \\ &= \frac{(2n+1)-1}{2}, && \text{since } 2n+1 \text{ is odd,} \\ &= n. \end{aligned}$$

In the second case, when $n < 0$ we get

$$\begin{aligned} f \circ g(n) &= f(g(n)) \\ &= f(-2n), && \text{since } n < 0, \\ &= -\frac{(-2n)}{2}, && \text{since } -2n \text{ is even,} \\ &= n. \end{aligned}$$

Thus $f \circ g(n) = n$ for all $n \in \mathbb{Z}$ and so $f \circ g = 1_{\mathbb{Z}}$.

Example 80 (c.f. Example 77(5))

The function

$$f : \mathbb{R} \to \mathbb{R},\ x \mapsto \frac{2x-1}{3}$$

has inverse

$$g : \mathbb{R} \to \mathbb{R},\ x \mapsto \frac{3x+1}{2}.$$

Check:

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{2x-1}{3}\right) = \frac{3\left(\frac{2x-1}{3}\right)+1}{2}$$

$$= \frac{(2x-1)+1}{2} = x = 1_{\mathbb{R}}(x)$$

and

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{3x+1}{2}\right) = \frac{2\left(\frac{3x+1}{2}\right)-1}{3}$$

$$= \frac{(3x+1)-1}{3} = x = 1_{\mathbb{R}}(x).$$