6.2 Functions

Definition

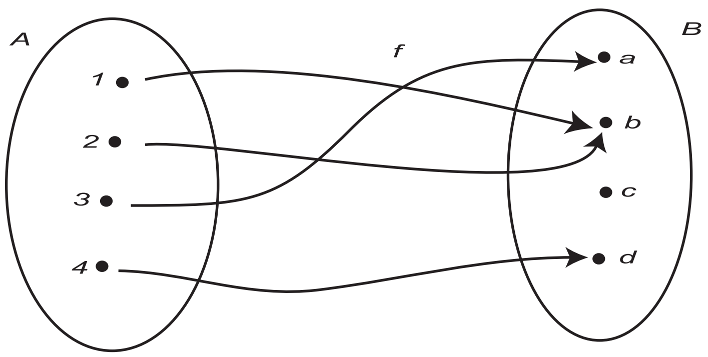
Let A and B be two sets (not necessarily different). Then a *function* is a rule that assigns to **every** element of A one **and only one** element of B. We let f denote this rule, and write $f : A \to B$.

If $a \in A$ and our rule assigns $b \in B$ to this a, we write $f(a) = b$. We say f is a function from A to B, A is the *domain* of f and B the *codomain*, and if $f(a) = b$ then b is the *image* of a.

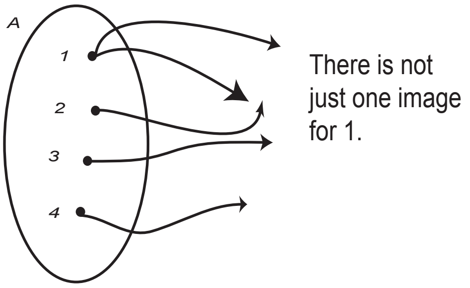
Note: A function depends on three things: the rule f, the domain A and codomain B. Change **any** of these and you have a different function.

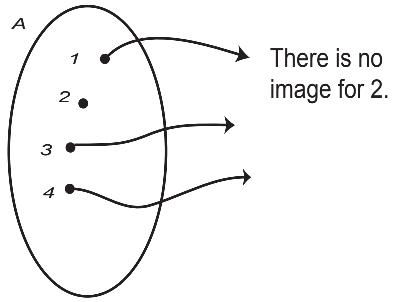
Describing Functions

(a) *Pictorial*
e.g.



Note we can use pictures to show what we do **not** want to see in a function. So we never want to see:





(b) *Relations*
Example 74 The above (pictorial) example may be written as

$$f = \{(1, b), (2, b), (3, a), (4, d)\} \subseteq A \times B.$$

So a function is a relation between sets. Yet $g = \{(1,b),(1,a),(3,b)\}$ is not a function because firstly: the image of 1 is not unique, and secondly: what are the images of 2 and 4?

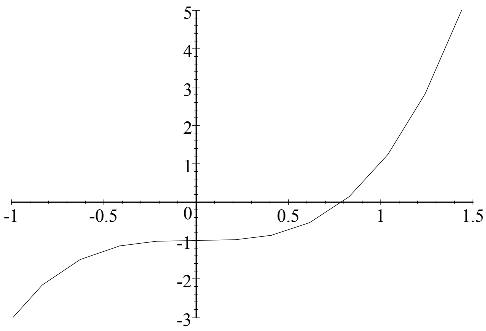
So functions are a *particular* type of relation between sets.

(c) *Formula*

Example 75 Define $f : \mathbb{R} \to \mathbb{R}$, by $f(x) = 2x^3 - 1$ (or $x \mapsto 2x^3 - 1$).

So $f(0) = -1,\ f(-1) = -3\ , f(1) = 1,\ f(2) = 15, ...$

Pictorially:



(d) *Inductively (*or *recursively)*

Example 76 Define $f : \mathbb{N} \to \mathbb{Q}$ by

$$f(1) = \frac{1}{2}, \qquad f(n+1) = 3f(n) + 5 \qquad \text{for all } n \geq 1.$$

So

$$f(1) = \frac{1}{2},\ f(2) = 3 \times \frac{1}{2} + 5 = \frac{13}{2},\ f(3) = 3 \times \frac{13}{2} + 5 = \frac{49}{2}, ...$$

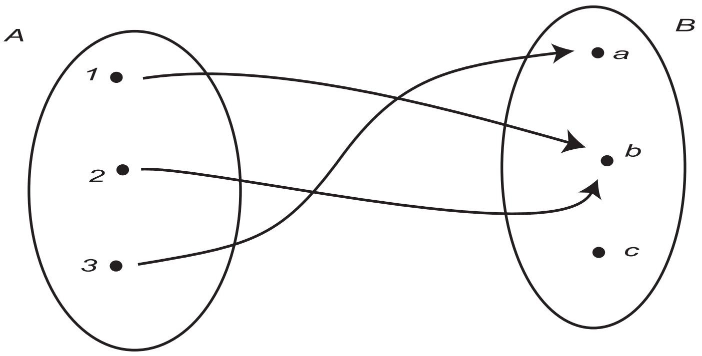
It can be shown that $f(n) = 3^n - (5/2)$ for all $n \geq 1$.

6.3 1–1 and Onto Functions

Definition

A function $f : A \to B$ is *one–to–one* (or *injective*) if different elements of A are mapped to different elements of B.

Pictorially, the definition of one-to-one means that we **never** want to see:



We can write out the definition using quantifiers because the definition tells us that we never want different elements to have the same image, that is

$$\begin{aligned}
&\neg(\exists a_1, \exists a_2, ((f(a_1) = f(a_2)) \wedge (a_1 \neq a_2)) \\
&\equiv \forall a_1, \neg\,(\exists a_2, ((f(a_1) = f(a_2)) \wedge (a_1 \neq a_2)) \\
&\equiv \forall a_1, \forall a_2, \neg((f(a_1) = f(a_2)) \wedge (a_1 \neq a_2)) \\
&\equiv \forall a_1, \forall a_2, ((f(a_1) = f(a_2)) \to \neg(a_1 \neq a_2)) \\
&\qquad\qquad\qquad\qquad \text{(recalling } \neg(p \wedge q) \equiv p \to \neg q) \\
&\equiv \forall a_1, \forall a_2, ((f(a_1) = f(a_2)) \to (a_1 = a_2)).
\end{aligned}$$

So, to check if a function is 1-1 we need show that the conditional statement

$$(f(a_1) = f(a_2)) \to (a_1 = a_2)$$

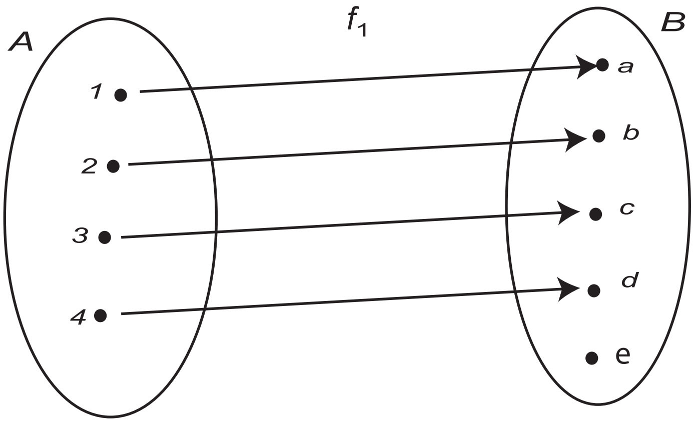
is true for all elements a_1 and a_2 from A. If we use the method of proof C.P. it suffices to assume $f(a_1) = f(a_2)$ and try to deduce $a_1 = a_2$.

Definition

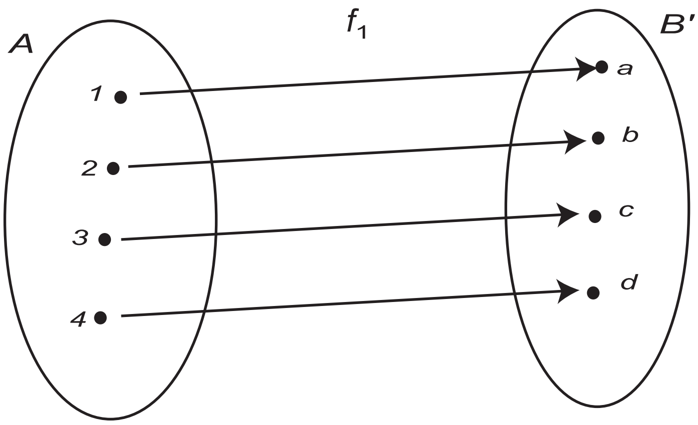
The function $f : A \to B$ is *onto* (or *surjective*) if each elements of B is the image of some element of A, i.e.

$$\forall b \in B,\ \exists a \in A :\ f(a) = b.$$

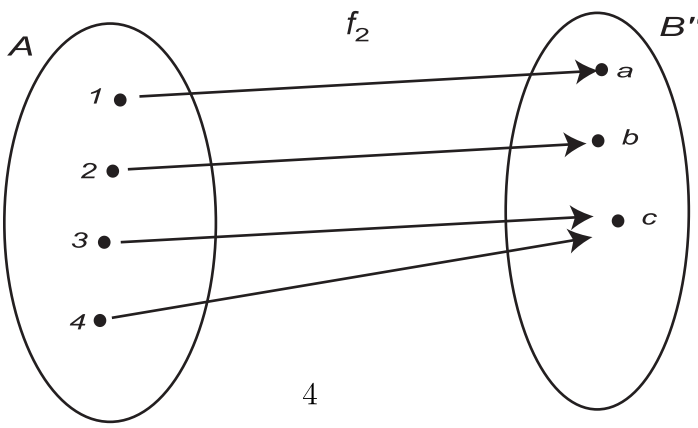
Example 77 (1)



is 1–1 but **not** onto.
(2)



is 1–1 and onto. So $f_1 : A \to B$ is a different function from $f_1 : A \to B'$, even though the rule is the same.
(3)



is **not** 1–1, because $3 \neq 4$ but $f_2(3) = f_2(4)$; yet the function is onto.

(4) The function $f : \mathbb{R} \to \mathbb{R},\ x \mapsto x^2$ is not 1–1 and is not onto.

Proof of (4) : We see that $-1 \neq 1$ but $f(-1) = 1 = f(1)$, so the function is not 1–1.

Now, given -1, there does not exist $y \in \mathbb{R}$ such that $f(y) = -1$, i.e. $y^2 = -1$ has no solution (in $\mathbb{R}$), so the function is not onto.

(5) The function $f : \mathbb{R} \to \mathbb{R},\ x \mapsto \frac{(2x-1)}{3}$ is 1–1 and onto.

Proof of (5): Assume $f(x) = f(y)$, so

$$\begin{aligned} & \tfrac{(2x-1)}{3} = \tfrac{(2y-1)}{3} \\ \times 3: \quad & 2x - 1 = 2y - 1 \\ +1: \quad & 2x = 2y \\ \times \tfrac{1}{2}: \quad & x = y. \end{aligned}$$

Thus $f(x) = f(y)$ implies $x = y$. Hence the function is 1–1.

Now, given any $x \in \mathbb{R}$ we need to find y such that f maps y onto x, that is,: $f(y) = x$, i.e. $\frac{2y-1}{3} = x$. Simply rearrange as

$$\begin{aligned} & \tfrac{2y-1}{3} = x \\ \times 3: \quad & 2y - 1 = 3x \\ +1: \quad & 2y = 3x + 1 \\ \times \tfrac{1}{2}: \quad & y = \tfrac{3x+1}{2}. \end{aligned}$$

So choose $y = \frac{3x+1}{2}$. As this works for any x, the function is onto.

Definition

We say that $f : A \to B$ is a *bijection* (or a *one–one correspondence*) if it is both 1–1 and onto.

So, by example 77(5) we see that $f : \mathbb{R} \to \mathbb{R},\ x \mapsto \frac{(2x-1)}{3}$ is a bijection.