# 6 Relations and Functions

## 6.1 Relations

**Definition**

Given two sets $A$ and $B$ (not necessarily different) a *relation from $A$ to $B$* is any subset $\mathcal{R} \subseteq A \times B$. If $(a, b) \in \mathcal{R}$ we write $a\mathcal{R}b$ and say that $a$ is *related* to $b$. If $a$ is *not* related to $b$, i.e. $(a, b) \notin \mathcal{R}$, write $aN\mathcal{R}b$.

If $A = B$, a relation *on* $A$ is any subset $\mathcal{R} \subseteq A \times A$.

**Example 66**

Let $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$.

(a) $\mathcal{R}_1 = \{(a, x), (b, z), (d, y), (c, x)\}$ is a relation from $A$ to $B$.

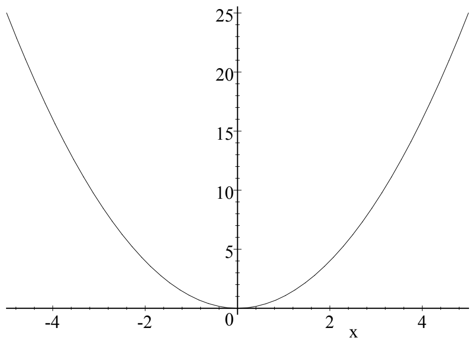
(b) $\mathcal{R}_2 = \{(x, d), (y, c)\}$ is a relation from $B$ to $A$.

(c) $\mathcal{R}_3 = \{(x, x), (x, y), (x, z)\}$ is a relation on $B$.

**Example 67**

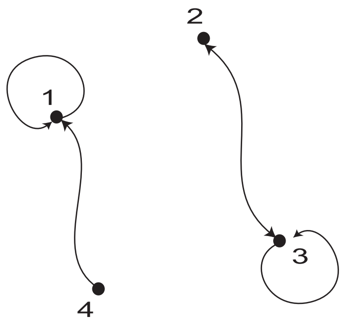
$\mathcal{R} = \{(x, x^2) : x \in \mathbb{R}\}$ is a relation on $\mathbb{R}$.

We can show this on a graph.

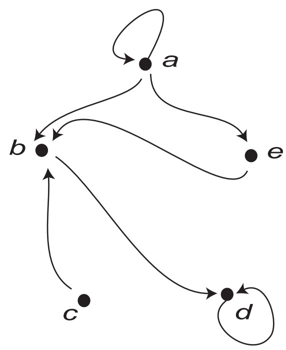



Consider $A = B$. Given $\mathcal{R} \subseteq A \times A$ we can denote it by a *directed graph* or *digraph* which consists of a set of *vertices* (or *nodes*) corresponding to elements of $A$, and *edges* (or *arcs*) that connect vertices $v$ and $w$ if, and only if, $(v, w) \in \mathcal{R}$ with an arrow pointing from $v$ to $w$.

**Example 68**
If $A = \{1, 2, 3, 4\}$ and $\mathcal{R} = \{(1, 1), (3, 2), (2, 3), (4, 1), (3, 3)\}$ this relation can be drawn as




**Example 69**
Starting with the digraph




we see that the relation on $\{a, b, c, d, e\}$ is

$$\{(a, a), (a, b), (a, e), (b, d), (c, b), (d, d), (e, b)\}.$$

**Special Properties**
A relation on a set $A$ may satisfy any of the following properties:

$\mathcal{R}$ is *reflexive*: for all $x \in A$, $(x, x) \in \mathcal{R}$, i.e.

$$\forall x : (x, x) \in \mathcal{R}.$$

$\mathcal{R}$ is *symmetric*: for all $x, y \in A$, if $(x, y) \in \mathcal{R}$ then $(y, x) \in \mathcal{R}$, i.e.

$$\forall x, \forall y : ((x, y) \in \mathcal{R}) \rightarrow ((y, x) \in \mathcal{R}).$$

$\mathcal{R}$ is *transitive*: for all $x, y, z \in A$, if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(x, z) \in \mathcal{R}$, i.e.

$$\forall x, \forall y, \forall z : (((x, y) \in \mathcal{R}) \wedge ((y, z) \in \mathcal{R})) \rightarrow ((x, z) \in \mathcal{R}).$$

For $\mathcal{R}$ to be reflexive means that in the digraph there is a loop on every vertex.

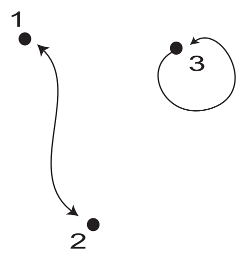
For $\mathcal{R}$ to be symmetric means that, in the digraph, on every path between *different* vertices there will be *two* arrows.

For $\mathcal{R}$ to be transitive you have to look at every example in the digraph of a path linking three vertices using two line. Then you have to check that there is one line linking the end points (i.e. you have to check that if you can go the 'long way round' then you can go the 'direct' way. Note that in the definition of transitive the vertices $x, y$ and $z$ need not be different.

**Example 70**

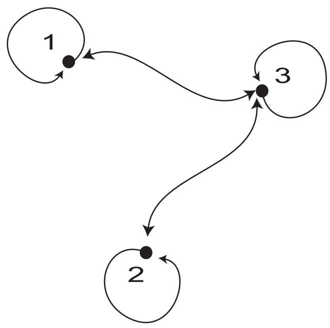
Let $A = \{1, 2, 3\}$.

(a) Let $\mathcal{R}_1 = \{(1, 2), (2, 1), (3, 3)\}$.



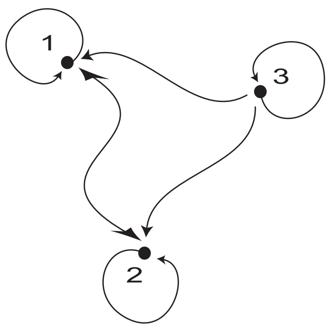

This relation is **not** reflexive, (there is no loop on 1, say) **is** symmetric, is **not** transitive $((1, 2), (2, 1) \in \mathcal{R}_1$ but $(1, 1) \notin \mathcal{R}_1)$.

(b) Let $\mathcal{R}_2 = \{(1, 3), (3, 1), (2, 3), (3, 2), (1, 1), (2, 2), (3, 3)\}$.




This relation **is** reflexive, **is** symmetric, is **not** transitive. $((1, 3), (3, 2) \in \mathcal{R}_2$ but $(1, 2) \notin \mathcal{R}_2)$

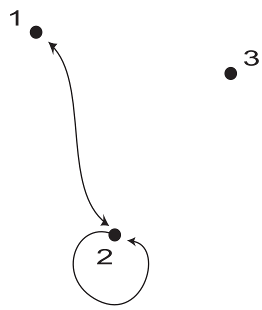
(c) Let $\mathcal{R}_3 = \{(1,2),(2,1),(3,1),(3,2),(1,1),(2,2),(3,3)\}$.




This relation **is** reflexive, is **not** symmetric ($(3,2) \in \mathcal{R}_3$ but $(2,3) \notin \mathcal{R}_3$), **is** transitive.

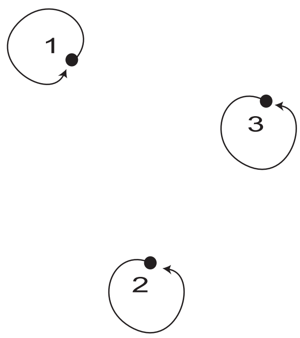
You have to be careful when checking transitivity.

(d) Let $\mathcal{R}_4 = \{(1,2),(2,1),(2,2)\}$.




This relation **is** reflexive, **is** symmetric, and is **not** transitive since $1R2$ but $2R1$ but $1NR1$.

(e) Let $\mathcal{R}_5 = \{(1,1),(2,2),(3,3)\}$.




This relation **is** reflexive, **is** symmetric, **is** transitive.

**Example 71**

Define $\mathcal{R}$ on $\mathbb{N}$ by $aRb$ if, and only if, $a < b$.

Then $R$ **is** transitive (e.g. $1 < 2$ and $2 < 3$ implies $1 < 3$) but $R$ is **not** reflexive (e.g. $1NR1$ since $1 \not< 1$) and $R$ is **not** symmetric (e.g. $2R3$ since $2 < 3$ but $3NR2$ since $3 \not< 2$).

**Definition** If $m, n \in \mathbb{Z}$ we say that $m$ *divides* $n$ if there exists $a \in \mathbb{Z}$ such that $n = ma$, and write $m|n$.

So for example 3 divides 6 since $6 = 3 \times 2$, and 3 divides $-6$ since $-6 = 3 \times (-2)$ , and also 3 divides 0 since $0 = 0 \times 3$. Of course all integers divide 0 for the same reason: $0 = 0 \times m$ for any $m \in \mathbb{Z}$.

**Example 72**

Define $\mathcal{R}$ on $\mathbb{Z}$ by $aRb$ if 3 divides $a - b$.

So, for example, $1R1$ (since 3 divides 0), and similarly $1R4, 1R7, 1R10, ...,$ while $1NR9$.

We show that $R$ satisfies all three properties.

Given $x \in \mathbb{Z}$, then by above 3 divides $x - x = 0$ so $xRx$, i.e. $\mathcal{R}$ **is** reflexive.

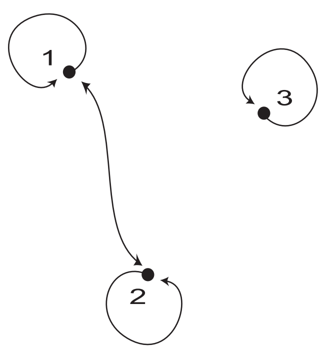
Given $x, y \in \mathbb{Z}$, if $xRy$ then 3 divides $x - y$ so 3 divides $-(x - y)$ i.e. 3 divides $y - x$, so $yRx$; i.e. $\mathcal{R}$ **is** symmetric.

Given $x, y, z \in \mathbb{Z}$, if $xRy$ and $yRz$ then 3 divides $x - y$ and $y - z$. By the definition this means we can find $a, b \in Z$ such that ,$x - y = 3a$ and $y - z = 3b$. Adding these two equations together we get $(x - y) + (y - z) = 3a + 3b$, that is $x - z = 3(a + b)$. Thus 3 divide. $x - z$ and so $xRz$, i.e. $\mathcal{R}$ **is** transitive.

**Definition**

A relation that satisfies all three properties (reflexive, symmetric and transitive) is called an *equivalence relation.*

**Example 73**




is an equivalence relation.