4 Quantifiers and Quantified Arguments

4.1 Quantifiers

Recall from Chapter 3 the definition of a predicate as an assertion containing one or more variables such that, if the variables are replaced by objects from a given Universal set U then we obtain a proposition.

Let $p(x)$ be a predicate with one variable.

Definition

If for all $x \in U$, $p(x)$ is true, we write $\forall x : p(x)$.
$\forall$ is called the *universal quantifier.*

Definition

If there exists $x \in U$ such that $p(x)$ is true, we write $\exists x : p(x)$.
$\exists$ is called the *existential quantifier.*

Note

(1) $\forall x : p(x)$ and $\exists x : p(x)$ are propositions and so, in any given example, we will be able to assign truth-values.

(2) The definitions can be applied to predicates with two or more variables. So if $p(x, y)$ has two variables we have, for instance, $\forall x, \exists y : p(x, y)$ if "for all x there exists y for which $p(x, y)$ is holds".

(3) Let A and B be two sets in a Universal set U. Recall the definition of $A \subseteq B$ as

"every element of A is in B."

This is a "for all" statement so we should be able to symbolize it. We do so by first rewriting the definition as

"for all $x \in U$, if x is in A then x is in B,"

which in symbols is

$$\forall x \in U : \text{ if } (x \in A) \text{ then } (x \in B),$$

or

$$\forall x : (x \in A) \rightarrow (x \in B).$$

(4*) Recall that a variable x in a propositional form $p(x)$ is said to be *free*. The variable x in $\forall x : p(x)$ or $\exists x : p(x)$ is said to be *bound*, that is, it is bound by the quantifier $\forall$ and $\exists$ respectively. In a statement of the form $\forall x : p(x, y)$ the variable x is bound while the variable y is free.

Example 43
Let $p(x)$ be the predicate "$x > 0$".

Then if $U = \mathbb{N} = \{1, 2, ...\}$ the proposition "$\forall x : x > 0$" is TRUE.

But if $U = \mathbb{Z} = \{..., -1, 0, 1, 2, ...\}$ the proposition "$\forall x, x > 0$" is FALSE.

So, the Universal set is important.

Example What are the truth values of the following propositions?

(i) $U = \mathbb{R}, \forall x :$ if $6 + x = 12$ then $5 + x = 11$.

Answer: True.

Reason: To be true we need that $(6 + x = 12) \to (5 + x = 11)$ is true for *every* possible choice of $x \in \mathbb{R}$. If we choose an x such that $6 + x = 12$ is False then, since $p \to q$ is True if p is False, we have that $(6 + x = 12) \to (5 + x = 11)$ is True. So the only possible problem is if we choose x such that $6+x = 12$ is True. This means we choose $x = 6$. But then $5+x = 5+6 = 11$, i.e. $5 + x = 11$ is True. So in this case $(6 + x = 12) \to (5 + x = 11)$ is True $\to$ True, which we know is true.

Hence in *all* cases, $(6 + x = 12) \to (5 + x = 11)$ is True.

(ii) $U = \mathbb{R}$, if $\forall x : 6 + x = 12$ then $\forall x : 5 + x = 11$.

Answer: True.

Reason: In symbols this is

$$(\forall x : 6 + x = 12) \to (\forall x : 5 + x = 11)\,. \tag{1}$$

But $\forall x : 6 + x = 12$ is False (it is easy to choose an $x \in \mathbb{R}$ such that $6 + x = 12$ is false, i.e. $x = 0$), hence (1) is True.

(iii) $U = \mathbb{R}, \forall x :$ if $\exists y : x + y = 0$ then $\exists y : xy = 0$.
Answer: True.
Reason: We have to answer the question of whether, given any $x \in \mathbb{R}$,

$$(\exists y : x + y = 0) \to (\exists y : xy = 0) \tag{2}$$

is True. But given x we can find y such that $x + y = 0$ (choose $y = -x$), so $\exists y : x + y = 0$ is True. Similarly, given x we can find y such that $xy = 0$ (choose $y = 0$), so $\exists y : xy = 0$ is True. Thus (2) is an example of True $\to$ True, which is True.

Example 44 Let $U = \mathbb{R}$.
Let $p(x, y)$ be the predicate "$x + y = 0$".

Given any $x \in \mathbb{R}$ choose $y = -x$ so that $x + y = 0$. Thus, for all x we can find a y such that $x + y = 0$, i.e. $\forall x \;\; \exists y : x + y = 0$ is TRUE.

Yet $\exists x \;\; \forall y : x + y = 0$ is FALSE for it says there exists an x such that for all y, $x + y = 0$. Just choose $y = 1 - x$ to see $x + y = 1 \neq 0$.

So $\forall x \;\; \exists y : p(x, y)$ need not be the same as $\exists x \;\; \forall y : p(x, y)$ and thus the *order* of the quantifiers is important.

Example 45 Let U = set of animals.
Symbolize "All cats eat meat".

Read this as: "For all $x \in U$, if x is a cat then x eats meat".
So if $Cx \equiv$ "x is a cat", $Mx \equiv$ "x eats meat", we get

$$\forall x : Cx \rightarrow Mx$$

Example 46 Let U = set of animals.
Symbolize "Some dogs drink wine".

Read this as: "There exists $x \in U$ such that x is a dog and x drinks wine".
So if $Dx \equiv$ "x is a dog", $Wx \equiv$ "x drinks wine", we get

$$\exists x : Dx \wedge Wx$$

Note (i) "Most times" a "For All" statement will have a $\rightarrow$ following the $\forall$, while a "There Exists" statement will have a $\wedge$ after the $\exists$.

(ii) The statment $\forall x : Cx \rightarrow Mx$ does not assert the existance of a cat in the universe of all animals, it just says that **if** a cat exists then it will have certain properties. On the other hand $\exists x : Dx \wedge Wx$ does assert that at least one of the animals in the universe is a dog, and you can in fact go further and find one of these dogs that drinks wine.

4.2 Negation

The rules are:

$$\neg\, (\forall x : \; p(x)) \equiv \exists x : \; \neg\, p(x)$$
$$\neg\, (\exists x : \; p(x)) \equiv \forall x : \; \neg\, p(x)$$

Example 47

$$\begin{aligned} \neg\,(\forall x : x > 0) &\equiv \exists x :\, \neg\,(x > 0) \\ &\equiv \exists x :\, x \leq 0 \end{aligned}$$

If $U = \mathbb{N}$ then $\exists x :\, x \leq 0$ is FALSE while if $U = \mathbb{Z}$ then it is TRUE. So we see that the negation of what was true in Example 43 is now false and visa-versa, as you would expect with negation.

Example 48 Let $U = \mathbb{R}$.

$$\begin{aligned} \neg\,(\forall x\ \ \exists y\ : x + y = 0) &\equiv \exists x : \neg\,(\exists y :\ x + y = 0) \\ &\equiv \exists x : \forall y \neg\,(x + y = 0) \\ &\equiv \exists x\ \forall y :\ x + y \neq 0 \end{aligned}$$

The last expression here is FALSE, for if such an x exists choose $y = -x$ to find an example when $x + y \neq 0$ does not hold.

Example 49 Let $U =$ set of animals and Cx, Mx be as in Example 45, then

$$\begin{aligned} \neg\,(\forall x : Cx \rightarrow Mx) &\equiv \exists x : \neg\,(Cx \rightarrow Mx) \\ &\equiv \exists x : Cx \wedge (\neg\, Mx) \end{aligned}$$

(Using $\neg\,(p \rightarrow q) \equiv p \wedge \neg q$ as seen in question Ex. 16.)

The final result says “Some cat does not eat meat” which is the negation of “All cats eat meat”.

Example 50 Let $U =$ set of animals and Dx, Wx be as in Example 45, then

$$\begin{aligned} \neg\,(\exists x,\ Dx \wedge Wx) &\equiv \forall x : \neg\,(Dx \wedge Wx) \\ &\equiv \forall x : Dx \rightarrow (\neg\, Wx) \end{aligned}$$

(Using $\neg\,(p \wedge q) \equiv p \rightarrow \neg q$ as can be checked by the student.)

The final results says “All dogs do not drink wine” which is the negation of “Some dogs drink wine”.

We see that negation fits in with the expectation noted earlier that $\forall$ is followed by a $\rightarrow$ while a $\exists$ is followed by a $\wedge$. For, as above,

$$\boxed{\begin{aligned} \neg\ (\exists x : px \wedge qx) &\equiv \forall x : px \rightarrow (\neg\, qx) \\ \neg\ (\forall x : px \rightarrow qx) &\equiv \exists x : px \wedge (\neg\, qx) \end{aligned}}$$

On the right-hand side we see that $\forall$ is followed by an $\rightarrow$ while $\exists$ is followed by an $\wedge$.

4.3 Arguments

We have the following rules:

U.S. Universal Specification

Given $\forall x : p(x)$ as a premise we can assume $p(u)$ for *any* $u \in U$.

E.S. Existential Specification

Given $\exists x : p(x)$ as a premise we can assume $p(u_0)$ for *some* $u_0 \in U$.

U.G. Universal Generalisation

If, for *any* $u \in U$, we can conclude $p(u)$ then we can conclude $\forall x : p(x)$.

E.G. Existential Generalisation

If, for *some* $u_0 \in U$, we can conclude $p(u_0)$ then we can conclude $\exists x : p(x)$.

Negation

If we have a step of the form $\neg\,(\forall x : p(x))$ we can conclude $\exists x : \neg\, p(x)$, and vice versa.

If we have a step of the form $\neg\,(\exists x : p(x))$ we can conclude $\forall x : \neg\, p(x)$, and vice versa.

Example 51 Let $U =$ the set of all people.
(i) Consider:

All maths lecturers have studied logic.
Dr. Coleman is a maths lecturer.
Therefore, Dr. Coleman has studied logic.

Let $m(x) \equiv$ "x is a maths lecturer", $l(x) \equiv$ "x has studied logic", and $u_0 \in U$ is the person called Dr. Coleman. Then the argument is symbolised as

$$\forall x\, : m(x) \to l(x), \quad m(u_0) \vdash l(u_0).$$

Here is a proof of validity:

1	$\forall x : m(x) \to l(x)$	A
2	$m(u_0) \to l(u_0)$	Universal Specification 1
3	$m(u_0)$	A
4	$l(u_0)$	MPP 2-3

Thus the argument is valid.

(ii) Consider:

All maths lecturers have studied logic.
Dr. Coleman is a maths lecturer.
Therefore, some person has studied logic.

This time the argument is symbolised as

$$\forall x\ : m(x) \rightarrow l(x), \quad m(u_0) \vdash \exists x : l(x).$$

Here is a proof of validity:

1	$\forall x :\ m(x) \rightarrow l(x)$	A
2	$m(u_0) \rightarrow l(u_0)$	Universal Specification 1
3	$m(u_0)$	A
4	$l(u_0)$	MPP 2-3
5	$\exists x : l(x)$	E.G. 4

Thus the argument is valid.

(iii) Consider:

All maths lecturers have studied logic.
Some person is a maths lecturer.
Therefore, some person has studied logic.

This time the argument is symbolised as

$$\forall x\ : m(x) \rightarrow l(x), \quad \exists x : m(x) \vdash \exists x : l(x).$$

Here is a proof of validity:

1	$\exists x : m(x)$	A
2	$m(u_0)$	E.S. 1
3	$\forall x :\ m(x) \rightarrow l(x)$	A
4	$m(u_0) \rightarrow l(u_0)$	Universal Specification 1
5	$l(u_0)$	MPP 2,4
6	$\exists x : l(x)$	E.G. 5

Thus the argument is valid.

The idea is to use the rules above to replace propositional forms containing quantifiers by forms without quantifiers on which we can use *any* and *all* of the rules of inference. Then, if the argument we are examining has a conclusion containing quantifiers, we must use the four rules again, because the rules of inference can only be applied to and have a conclusion of the form *not* containing quantifiers.

Example 52 Let U = set of all animals.

Consider:

Elephants have large ears,
All animals with large ears can hear well.
Therefore, elephants can hear well.

Let $Ex \equiv$ "x is an Elephant", $Lx \equiv$ "x has large ears", $Hx \equiv$ "x can hear well".

The first premise does not contain the word "All" but nonetheless we know that it is a statement about "All elephants". Thus the argument can be symbolized as

$$\forall x : Ex \to Lx, \;\; \forall x : Lx \to Hx \;\; \vdash \;\; \forall x : Ex \to Hx.$$

To prove $\forall x : Ex \to Hx$ we need to prove $Eu \to Hu$ for all $u \in U$. To prove $Eu \to Hu$ we use C.P.

(Ask yourself how you would have proved that the non-quantified argument, $e \to \ell, \;\; \ell \to h \;\vdash\; e \to \ell$, is valid. Not too difficult since we have seen this argument earlier in the course.)

1	⌈	Eu	A(C.P) and Specification to *any* $u \in U$,
2	|	$\forall x : Ex \to Lx$	A
3	|	$Eu \to Lu$	US 2
4	|	Lu	MPP 1,3
5	|	$\forall x : Lx \to Hx$	A
6	|	$Lu \to Hu$	US 5
7	⌊	Hu	MPP 4,6
8		$Eu \to Hu$	CP 1-7
9		$\forall x : Ex \to Hx$	UG 8

This method of proof is an extension of the Natural Deduction of section 2. We recall that Natural Deduction is a syntactic method, depending only on the form of the statements, not the meaning.

Is there a semantic method for predicate arguments? As we noted earlier it can be easier to prove a propositional argument is invalid by the semantic method rather than by the syntactic.

We cannot use truth tables but we can use the idea behind the truth-table method and say that a predicate argument is invalid if there is an *instance* of the argument where the premises are true yet the conclusion false. If, in all instances, this never happens we say the argument is valid.

The following has not been covered in 2004.

Example 53 Consider $\forall x : px \to qx, \quad qu_0 \vdash pu_0$ for some $u_0 \in U$ the universal set.

We look at an "instance" of this argument where px is replaced by $p \in P$ and qx by $x \in Q$ for two sets P and Q to be chosen.

The first premise now becomes $\forall x : (x \in P) \to (x \in Q)$ which is nothing other than the definition of set inclusion, i.e. $P \subseteq Q$. So in terms of sets our argument looks like

$$P \subseteq Q,\ u_0 \in Q\ \vdash\ u_0 \in P,$$

for some $u_0 \in U$. We now make particular choices of the sets. For instance $P = \{1, 2, 3, 4\}$ and $Q = \{1, 2, 3, 4, 5\}$ with $u_0 = 5$. With these choices the premises are both true yet the conclusion is false. We never want to see this in a valid argument. Hence our argument is not valid, i.e. it is invalid.

***Example** Consider $\exists x : px,\ \forall x : qx \to px\ \vdash\ \exists x : qx$.

If we are to replace px and qx by $p \in P$ and $x \in Q$ for two sets P and Q then the first premise says that $P \neq \emptyset$. So in terms of sets the argument looks like

$$P \neq \emptyset,\ Q \subseteq P\ \vdash\ Q \neq \emptyset.$$

But the choice of $P = \{1\}$ and $Q = \emptyset$ shows this argument to be invalid.

Further Examples

A more complicated example of the use of the rules of inference.

Example A

$$\forall x : p(x) \to (q(x) \vee r(x)),\ \forall x : \neg q(x),\ \exists x : \neg r(x)\ \vdash\ \exists x : \neg p(x).$$

We will use proof by contradiction, RAA. (Ask yourself how you would have proved that $p \to q \vee r$, $\neg q$, $\neg r \vdash \neg p$ is valid?)

1	$\lceil$	$\neg(\exists x : \neg p(x))$	A(RAA)
2	$\vert$	$\forall x : \neg(\neg(p(x)))$	negation
3	$\vert$	$\exists x : \neg r(x)$	A
4	$\vert$	$\neg r(u_0)$	E.S 3
5	$\vert$	$\neg(\neg(p(u_0)))$	U.S 2
6	$\vert$	$p(u_0)$	D.N 5
7	$\vert$	$\forall x : p(x) \to (q(x) \vee r(x))$	A
8	$\vert$	$p(u_0) \to (q(u_0) \vee r(u_0))$	U.S 7
9	$\vert$	$q(u_0) \vee r(u_0)$	M.P.P 6,8
10	$\vert$	$q(u_0)$	D.S 4,9
11	$\vert$	$\forall x : \neg q(x)$	A
12	$\vert$	$\neg q(u_0)$	U.S 11
13	$\lfloor$	$q(u_0) \wedge (\neg q(u_0))$	$\wedge$I, 10,12
14		$\exists x : \neg p(x)$	RAA 1-13.

A more complicated example of showing an argument is invalid is given in:

*__Example B__ Let U = Set of all people in the world.
Consider:

All Londoners live in England.
The President of France is not a Londoner.
Therefore, the President of France does not live in England.

All the propositions are true but does the conclusion "follow logically" from the premises?

To examine the "form" of the argument let

$Lx \equiv$ "x is a Londoner",
$Mx \equiv$ "x lives in England",
u_0 = The President of France $\in U$.

Then in symbolic form the argument reads:

$$\forall x : Lx \to Ex,\ \neg Lu_0 \ \vdash\ \neg Eu_0.$$

We look at another instance, for example as in Example 53 we look for an example using sets. So we interprete $Lx : x \in \mathcal{L}$ and $Ex : x \in \mathcal{E}$ for sets $\mathcal{L}$ and $\mathcal{E}$ to be chosen. Then the argument becomes

$$\mathcal{L} \subseteq \mathcal{E},\ u_0 \notin \mathcal{L} \ \vdash\ u_0 \notin \mathcal{E}$$

for some $u_0 \in U$. With the choice $\mathcal{L} = \{a, b, c, d\}$, $\mathcal{E} = \{a, b, c, d, e\}$ and $u_0 = e$ we find the premises are true while the conclusion is false. Hence the argument is invalid.

***Example C** Is $\exists x : px \wedge qx,\ \forall x : px \rightarrow rx \ \vdash\ \exists x : rx \wedge (\neg qx)$ valid or invalid?

So with U and $P, Q, R \subseteq U$ to be chosen we inteprete px as $p \in P$, qx as $q \in Q$ and rx as $r \in R$. This time $\exists x : px \wedge qx$ means there exists $u \in U$ such that $u \in P$ and $u \in Q$. In particular, $P \cap Q$ is non-empty, i.e. $P \cap Q \neq \emptyset$. So the argument becomes

$$P \cap Q \neq \emptyset,\ P \subseteq R \ \vdash\ R \cap Q^c \neq \emptyset.$$

The choice of $U = \{1, 2, 3\}$, $P = \{1\}$, $Q = \{1, 2\}$ and $R = \{1\}$ is an instance where the premises are true but the conclusion false. Hence the argument is invalid.