1 Propositional Logic

Propositions

1.1 Definition

A *declarative* sentence is a sentence that 'declares' a fact or facts.

Example 1

'The earth is spherical.'

'$7 + 1 = 6 + 2$'

'$x^2 > 0$ for all real numbers x.'

'$1 = 0$'

'This sentence is false'

These are all declarative sentences.

Definition

A *proposition* is a declarative sentence to which we can assign a truth-value of either *true* or *false*, but not both.

Example 2

'The earth is spherical.' is *true*.

'$7 + 1 = 6 + 2$' is *true*.

'$x^2 > 0$ for all real numbers x.' is *false* because $x = 0$ satisfies $x^2 = 0$ and not $x > 0$, so we have a *counterexample*.

'$1 = 0$' is *false*.

So these are all propositions.

*There are examples of declarative sentences that are not propositions. For example, 'This sentence is false' is not a proposition, since no truth value can be assigned. For instance, if we assign it the truth value True, then we are saying that 'This sentence is false' is a true fact, i.e. the sentence is false. Hence the sentence is simultaneously true and false. Similarly if we assign the truth value False to the sentence we get that the sentence is, again, simultaneously false and true. 'This sentence is false' is an example of a *paradox*, and gives an example of a declarative sentence that is not a proposition.

We will use letters such as $p, q, r, s, ...$ or $A, B, C, D, ...$ to represent propositions. The letters are called *logical variables*. We combine simple propositions to form compound propositions using connectives.

1.2.1 Connectives

"and", "or" and "not"

If p and q are propositions, then

(a) "$p \wedge q$" is the proposition "p and q", (the *conjunction* of p and q);
(b) "$p \vee q$" is the proposition "p or q or both", (the *disjunction* of p and q. It also known as the *inclusive or* . See Question Sheet 2, question 5)
Note: we normally omit the phrase "or both".
(c) "$\neg\, p$" is the proposition "not p".

Example 3

Let A denote "The earth is round"
Let B denote "The sun is cold"
Let C denote "It rains in Spain"

Then
(1) $A \wedge B$: "The earth is round and the sun is cold"
(2) $A \vee (\neg\, B)$: "Either the earth is round or the sun is not cold"
(3) $A \wedge (B \vee C)$: "The earth is round and either the sun is cold or it rains in Spain"

In the example A, B and C are propositions so we must be able to assign truth-values. In fact, here A is true (T), B is false (F) and C is true (T).

We demand that $A \wedge B$, $A \vee (\neg\, B)$ and $A \wedge (B \vee C)$ are propositions so we need to assign them truth-values. From the English sentences we see that $A \wedge B$ is false, $A \vee (\neg\, B)$ is true and $A \wedge (B \vee C)$ is true. But what if A were false, B false and C true?

1.2.2 Truth Tables

Definition

A *propositional form* is an expression involving logical variables and connectives such that, if **all** the variables are replaced by propositions then the form becomes a proposition.

Example 4

$p \wedge (q \vee r)$ is a propositional form with variables p, q and r.

If we set $p =$"$2^2 > 3$", $q =$"$3^2 > 8$" and $r =$"$2 \times 3 > 8$" then $p \wedge ((\neg\, q) \vee r)$ becomes

$$\text{“}2^2 > 3 \text{ and either } 3^2 > 8 \text{ or } 2 \times 3 > 8\text{”},$$

which is a proposition.

*CAREFUL If we miss just one word we might write

$$\text{“}2^2 > 3 \text{ and } 3^2 > 8 \text{ or } 2 \times 3 > 8\text{”}. \tag{1}$$

Someone might well read this as

$$\text{“either } 2^2 > 3 \text{ and } 3^2 > 8 \text{ or } 2 \times 3 > 8\text{”}$$

and so symbolise it as $(p \wedge q) \vee r$. We will see later that

$$p \wedge (q \vee r) \text{ and } (p \wedge q) \vee r$$

are different forms. So (1) is ambiguous. Always, always make sure that your statements are unambiguous.

We use Truth Tables to assign truth-values to propositional forms when truth-values have been assigned to the variables in the form. So we have:

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

* **Note** In this course you must use T and F, not 1 and 0 as you might have learnt at school. Also, the rows must always be in the order I have them, ie TT, TF, FT etc or TTT, TTF, TFT, etc. If you have them different I may well accidently mark your work wrongly.

Example 5

If A and B are false and C is true then $A \wedge (B \vee C)$ has a truth-value that can be represented by $F \wedge (F \vee T)$. From the third table, third row, we see that we can write this as $F \wedge T$ and then, by using table two, third row, we can simplify this to F.
Hence $A \wedge (B \vee C)$ is false.

Example 6

The truth table for the propositional form $((\neg P) \vee Q) \wedge (\neg R)$ is

P	Q	R	$(\neg P)$	$(\neg P) \vee Q$	$(\neg R)$	$((\neg P) \vee Q) \wedge (\neg R)$
T	T	T	F	T	F	F
T	T	F	F	T	T	T
T	F	T	F	F	F	F
T	F	F	F	F	T	F
F	T	T	T	T	F	F
F	T	F	T	T	T	T
F	F	T	T	T	F	F
F	F	F	T	T	T	T

Definition

Two propositional forms are *equivalent* if they have the same truth–table.

Example 7

p	q	$p \wedge q$	$\neg\,(p \wedge q)$	$(\neg\, p)$	$(\neg\, q)$	$(\neg\, p) \vee (\neg\, q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T
			*			*

We see from the two starred columns above that $\neg\,(p \wedge q)$ and $(\neg\, p) \vee (\neg\, q)$ are equivalent. We write $\neg\,(p \wedge q) \equiv (\neg\, p) \vee (\neg\, q)$.

This is one of the two

De Morgan's laws for propositions:

(a) $\neg\,(p \wedge q) \equiv (\neg\, p) \vee (\neg\, q)$
(b) $\neg\,(p \vee q) \equiv (\neg\, p) \wedge (\neg\, q)$

Part (b) is proved using truth-tables as well.

Note, in English, the construction $\neg\,(p \vee q)$ is often written as "neither p nor q hold".

Definition

A propositional form that is always true is called a *tautology*.

Example 8

Consider $(p \vee q) \vee ((\neg\, p) \wedge (\neg\, q))$:

p	q	$\neg\, p$	$\neg\, q$	$p \vee q$	$\neg\,(p) \wedge (\neg\, q))$	$(p \vee q) \vee ((\neg\, p) \wedge (\neg\, q))$
T	T	F	F	T	F	T
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	F	T	T

So $(p \vee q) \vee ((\neg\, p) \wedge (\neg\, q))$ is a tautology.

Definition

A propositional form that is always false is called a *contradiction.*

Example 9

Consider $(p \wedge q) \wedge ((\neg p) \vee (\neg q))$:

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg (p) \vee (\neg q))$	$(p \wedge q) \wedge ((\neg p) \vee (\neg q))$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	F

So $(p \wedge q) \wedge ((\neg p) \vee (\neg q))$ is a contradiction.

We will use I to denote any tautology and O to denote any contradiction.

So, since

p	$\neg p$	$p \vee (\neg p)$
T	F	T
F	T	T

,

we have that $p \vee (\neg p)$ is a tautology and we write $p \vee (\neg p) \equiv I$.

Similarly,

p	$\neg p$	$p \wedge (\neg p)$
T	F	F
F	T	F

,

So $p \wedge (\neg p)$ is a contradiction and we write $p \wedge (\neg p) \equiv O$.

These are just two of the results in

The Boolean Laws of Logic:

1(a) $p \vee q \equiv q \vee p$
1(b) $p \wedge q \equiv q \wedge p$ } (Commutative laws)

2(a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$
2(b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ } (Associative laws)

3(a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
3(b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ } (Distributive laws)

4(a) $p \vee 0 \equiv p$
4(b) $p \wedge I \equiv p$
5(a) $p \vee (\neg p) \equiv I$
5(b) $p \wedge (\neg p) \equiv 0$

* We consider De Morgans laws to be included in the Boolean laws of logic.

All these results can be proved using truth-tables.

Example 10

p	q	r	$q \wedge r$	$p \vee q$	$p \vee r$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	T	F	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

The last two columns are identical so we have the first distributive law, 3a.

* **Note** In

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \tag{2}$$

the p, q and r are simply labels of propositions. If the propositions are relabeled we still have an equivalence. For instance, if we replace p by A, q by B and r by C we get

$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C),$$

which is a correct statement. But further, there is no saying that these labels might not, themselves, be complicated statements. So, for example, if we carry on and replace A by $(p \wedge q)$, B by r and C by s we get

$$(p \vee q) \vee (r \wedge s) \equiv ((p \vee q) \vee r) \wedge ((p \vee q) \vee s)\,.$$

This may look more complicated than (2), for instance it has four variables in place of three, but it is nonetheless still simply a form of the distributive law.

Once we have proved the Boolean laws we can use them to simplify expressions.

Example 11

$$((\neg r) \wedge p \wedge q) \vee ((\neg r) \wedge (\neg p) \wedge q)$$

$$\begin{aligned}
&((\neg r) \wedge (p \wedge q)) \vee ((\neg r) \wedge ((\neg p) \wedge q)) & \\
&\equiv (\neg r) \wedge ((p \wedge q) \vee ((\neg p) \wedge q)) & \text{law 3b} \\
&\equiv (\neg r) \wedge ((q \wedge (\neg p)) \vee (q \wedge (\neg p))) & \text{law 1b} \\
&\equiv (\neg r) \wedge (q \wedge (p \vee (\neg p))) & \text{law 3b} \\
&\equiv (\neg r) \wedge (q \wedge I) & \text{law 5a} \\
&\equiv (\neg r) \wedge q & \text{law 4b}
\end{aligned}$$

*Additional Material (Not covered in lectures)

It might be noticed that some results which are obvious from truth tables are not listed in the Boolean Laws of logic. For example it is easy to show using truth tables that we have the following two results.

$$\left.\begin{array}{l} p \vee I \equiv I \\ p \wedge O \equiv O \end{array}\right\} \qquad \text{(Domination Laws)}$$

Using the Boolean laws along with De Morgan's laws we can prove these results. For example:

$$\begin{array}{lll} O & \equiv p \wedge \neg p & \text{law 5b} \\ & \equiv p \wedge (\neg p \vee O) & \text{law 4a} \\ & \equiv (p \wedge \neg p) \vee (p \wedge O) & \text{law 3b} \\ & \equiv O \vee (p \wedge O) & \text{law 5b} \\ & \equiv p \wedge O & \text{law 4a.} \end{array}$$

Similarly we can prove the next two results

$$\left.\begin{array}{l} p \vee p \equiv p \\ p \wedge p \equiv p \end{array}\right\} \qquad \text{(Idempotent Laws)}$$

But maybe the simplest result is

$$\neg(\neg p) \equiv p \qquad \text{(Double negative)}$$

Can we deduce this from the Boolean laws? Yes, but it isn't straightforward and will **not** be examined.

$$\begin{array}{lll} \neg(\neg p) & \equiv (\neg(\neg p)) \wedge I & \text{law 4b} \\ & \equiv (\neg(\neg p)) \wedge (\neg p \vee p) & \text{law 5a (and 1a)} \\ & \equiv ((\neg(\neg p)) \wedge \neg p) \vee ((\neg(\neg p)) \wedge p) & \text{law 3b} \\ & \equiv O \vee ((\neg(\neg p)) \wedge p) & \text{law 5b (and 1b)} \\ & \equiv (\neg p \wedge p) \vee ((\neg(\neg p)) \wedge p) & \text{law 5b (and 1b)} \\ & \equiv (p \wedge \neg p) \vee (p \wedge (\neg(\neg p))) & \text{law 1b} \\ & \equiv p \wedge ((\neg p) \vee (\neg(\neg p))) & \text{law 3b} \\ & \equiv p \wedge I & \text{law 5a} \\ & \equiv p & \text{law 4b.} \end{array}$$