Symbolising Quantified Arguments

1. (i) Symbolise the following argument, given the universe of discourse is U = set of all animals.

Animals are either male or female.
Not all Cats are male,
Therefore, some cats are female.

Let $Cx = x$ is a cat, $Mx = x$ is male and $Fx = x$ is female.

$$\forall x : Mx \vee Fx,\ \neg\left(\forall x : Cx \rightarrow Mx\right)\ \vdash\ \exists x : Cx \wedge Fx.$$

(ii) U = set of all animals.

Not all animals are cats,
An Animal must be a cat if it has a tail,
Therefore, not all animals have tails.

Let $Cx = x$ is a cat and $Tx = x$ has a tail.

$$\neg\left(\forall x : Cx\right),\ \forall x : Tx \rightarrow Cx\ \vdash\ \neg\left(\forall x : Tx\right).$$

(iii) U = set of all animals.

All animals are either male or female,
Tom is not female,
Therefore, Tom is male.

Let $Mx = x$ is male, $Fx = x$ is female, $t \in U$ is the animal known as Tom (we assume this is unique)

$$\forall x : Mx \vee Fx,\ \neg Ft\ \vdash\ Mt.$$

(iv) Symbolise the following argument, given the universe of discourse is U = set of all elephants.

Elephants are either pink or grey,
All pink elephants can fly,
No elephants can fly,
Therefore, all elephants are grey.

Let $Px = x$ is pink, $Gx = x$ is grey and $Fx = x$ can fly.

$$\forall x : Px \vee Gx,\ \forall x : Px \rightarrow Fx,\ \neg\left(\exists x : Fx\right)\ \vdash\ \forall x : Gx.$$

(v) Repeat part (i) but with the Universe replaced by U = set of all animals.

Let $Ex = x$ is an elephant, along with $Px = x$ is pink, $Gx = x$ is grey and $Fx = x$ can fly.

$$\begin{aligned}\forall x \ &: \ Ex \to (Px \vee Gx)\,,\ \forall x : (Px \wedge Ex) \to Fx,\ \neg\,(\exists x : Ex \wedge Fx) \\ &\vdash \ \forall x : Ex \to Gx.\end{aligned}$$

Choosing an appropriate Universe will often simplify the symbolising of an argument. But remember to choose the Universe to contain all objects to which the argument refers.

(vi) What would be appropriate universe for each of the following?

(a) Animals are either male or female, *Set of all animals.*

(b) All cats eat meat, *Set of all cats.*

(c) Some cats eat mice. *Set of all cats*

(d) Some mice are eaten by cats. *Set of all mice.*

(e) Some mice are eaten by cats while some cats are nibbled by mice. *Set of all cats and all mice*, or perhaps the *set of all animals.*

Note In (c), if U is the set of all cats, $Cx = x$ is a cat and $mx = x$ eats mice, the sentence becomes $\exists x : Cx \wedge mx$. But we could change the universe to U, the set of all cats and all mice (or even the set of all animals). This time we need a predicate with two variables, $e(x, y) = x$ eats y, along with $Cx = x$ is a cat and $Mx = x$ is a mouse. We then symbolise (c) as $\exists x, \exists y : Cx \wedge My \wedge e(x, y)$. How would (d) be rewritten with this larger universe?

(vi) $U =$ set of all animals.

If some cat is male then some dog is female.
Tom is a male cat.
Therefore, not all dogs are male.

Let $Cx = x$ is a cat, $Dx = x$ is a dog, $Mx = x$ is male, $Fx = x$ is female and $t \in U$ is the animal known as Tom.

$$(\exists x : Cx \wedge Mx) \to (\exists x : Dx \wedge Fx)\,, Mt \ \vdash \ \neg\,(\forall x : Dx \to Mx)\,.$$

Proving Quantified Arguments

1) Give proofs of validity for the following arguments.

(i) $\forall x : Mx \vee Fx,\ \neg(\forall x : Cx \to Mx)\ \vdash\ \exists x : Cx \wedge Fx.$

1	$\neg(\forall x : Cx \to Mx)$	A
2	$\exists x : \neg(Cx \to Mx)$	Negation
3	$\neg(Cu_0 \to Mu_0)$	ES 2, for some $u_0 \in U$,
4	$Cu_0 \wedge \neg Mu_0$	?
5	$\forall x : Mx \vee Fx$	A
6	$Mu_0 \vee Fu_0$	US 5
7	$\neg Mu_0$	$\wedge$E 4
8	Fu_0	DS 6,7
9	Cu_0	$\wedge$E 4
10	$Cu_0 \wedge Fu_0$	$\wedge$I 8,9
11	$\exists x : Cx \wedge Fx$	EG 10

There is a problem in this proof. We have not justified step 4, shown here as a ?. We can though, fit in the proof of $\neg(p \to q) \equiv p \wedge (\neg q)$ found in Additional Question 3(9). This would, though, lead to a very long proof.

(ii) $\forall x : \neg p(x)\ \vdash\ \neg(\exists x : p(x))$

1	⌈	$\neg(\neg(\exists x : p(x)))$	A(RAA)
2	|	$\exists x : p(x)$	DN 1
3	|	$p(u_0)$	ES 2, some $u_0 \in U$
4	|	$\forall x : \neg p(x)$	A
5	|	$\neg p(u_0)$	US 4
6	⌊	$p(u_0) \wedge (\neg p(u_0))$	I$\wedge$ 3,5
7		$\neg(\exists x : p(x))$	RAA 1-6

(iii) $\neg(\forall x : Cx),\ \forall x : Tx \to Cx\ \vdash\ \neg(\forall x : Tx),$

1	⌈	$\neg(\neg(\forall x : Tx))$	A(RAA)
2	|	$\forall x : Tx$	DN 1
3	|	Tu	US 2 any $u \in U$,
4	|	$\forall x : Tx \to Cx$	A
5	|	$Tu \to Cu$	US 4
6	|	Cu	MPP 3,5
7	|	$\forall x : Cx$	UG 6
8	|	$\neg(\forall x : Cx)$	A
9	⌊	$(\forall x : Cx) \wedge (\neg(\forall x : Cx))$	$\wedge$I 7,8
10		$\neg(\forall x : Tx)$	RAA 1-9

(iv) $\forall x : Px \vee Gx,\ \forall x : Px \rightarrow Fx,\ \neg(\exists x : Fx)\ \vdash\ \forall x : Gx.$

1	$\neg(\exists x : Fx)$	A
2	$\forall x : \neg Fx$	Negation 1
3	$\neg Fu$	US 2 any $u \in U$
4	$\forall x : Px \rightarrow Fx$	A
5	$Pu \rightarrow Fu$	US 4
6	$\neg Pu$	MTT 3,5
7	$\forall x : Px \vee Gx$	A
8	$Pu \vee Gu$	US 7
9	Gu	DS 6,8
10	$\forall x : Gx$	UG 9

2) Give proofs of validity for the following.

(i) $\exists x : p(x) \vee q(x)\ \vdash\ (\exists x : p(x)) \vee (\exists x : q(x))$,

1		$\exists x : p(x) \vee q(x)$	A
2		$p(u_0) \vee q(u_0)$	ES 1, some $u_0 \in U$
3	⌈	$p(u_0)$	$\vee$E 2
4	|	$\exists x : p(x)$	EG 3
5	⌊	$(\exists x : p(x)) \vee (\exists x : q(x))$	$\vee$I 4
6	⌈	$q(u_0)$	$\vee$E 2
7	|	$\exists x : q(x)$	EG 6
8	⌊	$(\exists x : p(x)) \vee (\exists x : q(x))$	$\vee$I 7
9		$(\exists x : p(x)) \vee (\exists x : q(x))$	$\vee$E 3-8

(ii) $(\exists x : p(x)) \vee (\exists x : q(x))\ \vdash\ \exists x : p(x) \vee q(x)$,

1		$(\exists x : p(x)) \vee (\exists x : q(x))$	A
2	⌈	$\exists x : p(x)$	$\vee$E 1
3	|	$p(u_0)$	ES 2, some $u_0 \in U$,
4	|	$p(u_0) \vee q(u_0)$	$\vee$I 3
5	⌊	$\exists x : p(x) \vee q(x)$	EG 4
6	⌈	$\exists x : q(x)$	$\vee$E 1
7	|	$q(u_1)$	ES 6, some $u_1 \in U$,
8	|	$p(u_1) \vee q(u_1)$	$\vee$I 7
9	⌊	$\exists x : p(x) \vee q(x)$	EG 8
10		$\exists x : p(x) \vee q(x)$	$\vee$E 2-9

Have used different symbols u_0 and u_1 since $p(x)$ and $q(x)$ might be true for different objects in the universe.

(iii) $(\forall x : p(x)) \vee (\forall x : q(x)) \;\vdash\; \forall x : p(x) \vee q(x)$,

1		$(\forall x : p(x)) \vee (\forall x : q(x))$	A
2	$\lceil$	$\forall x : p(x)$	$\vee$E 1
3	$\vert$	$p(u)$	US 2 any $u \in U$
4	$\vert$	$p(u) \vee q(u)$	$\vee$I 3
5	$\lfloor$	$\forall x : p(x) \vee q(x)$	UG 4
6	$\lceil$	$\forall x : q(x)$	$\vee$E 1
7	$\vert$	$q(v)$	US 6 any $v \in U$
8	$\vert$	$p(v) \vee q(v)$	$\vee$I 7
9	$\lfloor$	$\forall x : p(x) \vee q(x)$	UG 8
10		$\forall x : p(x) \vee q(x)$	$\vee$E 2-9

(iv) $\forall x : p(x) \to q(x)$, $\neg(\exists x : q(x)) \;\vdash\; \neg(\exists x : p(x))$,

1	$\neg(\exists x : q(x))$	A
2	$\forall x : \neg q(x)$	Negation 1
3	$\neg q(u)$	US 2 any $u \in U$
4	$\forall x : p(x) \to q(x)$	A
5	$p(u) \to q(u)$	US 4
6	$\neg p(u)$	MTT 3,5
7	$\forall x : \neg p(x)$	UG 7
8	$\neg(\exists x : p(x))$	Negation 7

(v) $(\exists x : p(x)) \to (\exists x : q(x))$, $\exists x : p(x) \;\vdash\; \exists x : q(x)$

TRICK: This is not really a predicate logic problem, merely a propositional logic problem.

1	$\exists x : p(x)$	A
2	$(\exists x : p(x)) \to (\exists x : q(x))$	A
3	$\exists x : q(x)$	MPP 1,2

(vi) $\forall w : pw \to qw \;\vdash\; (\forall x : qx \to rx) \to (\forall y : py \to ry)$,

Note, the bound variables do not have to be x, and don't have to be the same throughout an argument.

1	$\forall x : qx \to rx$	A(CP)
2	pu	A(CP) any $u \in U$
3	$\forall w : pw \to qw$	A
4	$pu \to qu$	US 3
5	qu	MPP 2,4
6	$qu \to ru$	US 1
7	ru	MPP 5,6
8	$pu \to ru$	CP 2-7
9	$\forall y : py \to ry$	UG 8
10	$(\forall x : qx \to rx) \to (\forall y : py \to ry)$	CP 1-9

(vi) $\forall x : ax \to (bx \vee rx)\,,\ \neg\,(\exists x : rx)\ \vdash\ \forall x : ax \to bx.$

1	au	A(CP) any u
2	$\forall x : ax \to (bx \vee rx)$	A
3	$au \to (bu \vee ru)$	US 2
4	$bu \vee ru$	MPP 1,3
5	$\neg\,(\exists x : rx)$	A
6	$\forall x : \neg rx$	Negation 5
7	$\neg ru$	US 6
8	bu	DS 4,7
9	$au \to bu$	CP 1-8
10	$\forall x : ax \to bx.$	UG 9

(vii) $\neg\,(\exists x : p(x) \vee q\,(x))\ \vdash\ \forall x : \neg p(x).$

1	$\neg\,(\forall x : \neg p(x))$	A(RAA)
2	$\exists x : \neg\,(\neg p(x))$	Negation 1
3	$\neg\,(\neg p(u_0))$	ES 2, some $u_0 \in U$
4	$p\,(u_0)$	DN 3
5	$p\,(u_0) \vee q\,(u_0)$	$\vee$I 4
6	$\exists x : p(x) \vee q\,(x)$	EG
7	$\neg\,(\exists x : p(x) \vee q\,(x))$	A
8	$(\exists x : p(x) \vee q\,(x)) \wedge (\neg\,(\exists x : p(x) \vee q\,(x)))$	$\wedge$I 6,7
9	$\forall x : \neg p(x)$	RAA 1-8

(viii)

$$\begin{aligned}\forall x, \exists y &: p(x,y) \rightarrow r(x,y),\ \exists x, \forall y : r(x,y) \rightarrow s(x,y), \\ \forall x, \forall y &: p(x,y)\ \vdash\ \exists x, \exists y : s(x,y)\end{aligned}$$

1	$\exists x, \forall y : r(x,y) \rightarrow s(x,y)$	A
2	$\forall y : r(u_0,y) \rightarrow s(u_0,y)$	ES 1 some $u_0 \in U$
3	$\forall x, \exists y : p(x,y) \rightarrow r(x,y)$	A
4	$\exists y : p(u_0,y) \rightarrow r(u_0,y)$	US 3
5	$p(u_0,v_1) \rightarrow r(u_0,v_1)$	ES 4 some $\nu_1 \in U$
6	$\forall x, \forall y : p(x,y)$	A
7	$\forall y : p(u_0,y)$	US 6
8	$p(u_0,v_1)$	US 7
9	$r(u_0,v_1)$	MPP 5,8
10	$r(u_0,v_1) \rightarrow s(u_0,v_1)$	US 2
11	$s(u_0,v_1)$	MPP 9,10
12	$\exists y : s(u_0,y)$	EG 11
13	$\exists x, \exists y : s(x,y)$	EG 12

Invalid arguments

Show that the following arguments are invalid.

1) $\forall x : p(x) \vee q(x)\ \vdash\ (\forall x : p(x)) \vee (\forall x : q(x))$

In set form this argument reads $P \cup Q = U\ \vdash\ (P = U) \vee (Q = U)$. It should be obvious to a student how to construct a counter-example.

2) $\forall x : p(x)\ \vdash\ \exists x : p(x)$,

In sets: $P = U\ \vdash\ P \neq \emptyset$,

Choose $U = \emptyset$ and $P = U$ when the premise will be true but the conclusion false. This may not be what you expect, but shows a difference between the quantifiers. To say that $\exists x : p(x)$ is true is to say there *exists* an object with a certain property. To say $\forall x : p(x)$ is true is not to assert that any object exists, but rather, if objects exist then they will have a certain property.

3) $\forall x : p(x) \rightarrow r(x) \vee s(x),\ \exists x : r(x)\ \vdash\ \exists x : s(x) \vee p(x)$.

In sets: $P \subseteq R \cup S,\ R \neq \emptyset\ \vdash\ S \cup P \neq \emptyset$,

Take $U = \{1\}, R = \{1\}$ and $P = S = \emptyset$.

4) $(\exists x : p(x)) \to (\exists x : q(x))\,,\ \forall x : q(x)\ \vdash\ \exists x : p(x)$

In sets: $(P \neq \emptyset) \to (Q \neq \emptyset)\,,\ Q = U\ \vdash\ P \neq \emptyset$,

Take $U = \{1\} = Q$ and $P = \emptyset$.

5) $(\forall x : p(x) \to q(x)) \to (\exists x : p(x))\,,\ (\forall x : q(x) \to p(x)) \to (\exists x : q(x))$
$\vdash\ \exists x : p(x) \wedge q(x)\,,$

In sets: $(P \subseteq Q) \to (P \neq \emptyset)\,,\ (Q \subseteq P) \to (Q \neq \emptyset)\ \vdash\ P \cap Q \neq \emptyset$,

Take $U = \{1, 2\}$, $P = \{1\}$ and $Q = \{2\}$.

6) $\forall x : p(x) \to r(x) \vee s(x)\,,\ \neg ru_0,\ \exists x : p(x)\ \vdash\ su_0.$

In sets: $P \subseteq (R \cup S)\,,\ u_0 \notin R,\ P \neq \emptyset\ \vdash\ u_0 \in S$.

Take $U = \{1, 2, 3, 4\}$, $P = \{1\}\,,\ R = \{1, 2\}\,,\ S = \{3\}$ and $u_0 = 4$.

7) Let $U =$ set of all animals.

(i) The only animals with tails are cats.
Tom is a cat.
Therefore, Tom has a tail.

Let $Cx = x$ is a cat, $Tx = x$ has a tail and $t \in U$ be the animal known as Tom.

The argument becomes $\forall x : Tx \to Cx,\ Ct\ \vdash\ Tx$.

In terms of sets: $T \subseteq C, t \in C\ \vdash\ t \in T$.

Take $U = \{1, 2\}\,, T = \{1\}\,, C = \{1, 2\}$ and $t = 2$.

(ii) All animals have tails if they are cats,
Jerry is not a cat,
Therefore, Jerry does not have a tail.

Let $j \in U$ be the animal known as Jerry.

The argument becomes $\forall x : Cx \to Tx,\ \neg Cj\ \vdash\ \neg Tj$.

In sets: $C \subseteq T,\ j \notin C\ \vdash\ j \notin T$.

Take $U = \{1, 2\}\,, C = \{1\}\,, T = \{1, 2\}$ and $j = 2$.

(iii) If cats exist then dogs exist.
Not all animals are dogs,
Therefore, some animals are not cats.

The argument becomes:

$(\exists x : Cx) \to (\exists x : Dx)\,,\ \neg(\forall x : Dx)\ \vdash\ \exists x : \neg Cx$.

In sets: $(C \neq \emptyset) \to (D \neq \emptyset)\,,\ D \neq U\ \vdash\ C^c \neq \emptyset$.

Take $U = \{1\}$ and $C = D = \emptyset$.