Section 5 Series with non-negative terms

Theorem 5.1 Let $\sum_{r=1}^{\infty} a_r$ be a series with non-negative terms and let s_n be the n-th partial sum for each $n \in \mathbb{N}$. Then $\sum_{r=1}^{\infty} a_r$ is convergent if, and only if, $\{s_n\}_{n\in\mathbb{N}}$ is bounded.

Proof Since $a_r \geq 0$ for all $r \in \mathbb{N}$, then $s_{n+1} - s_n = a_{n+1} \geq 0$, i.e. $s_{n+1} \geq s_n$ for all $n \geq 1$ and so the sequence $\{s_n\}_{n\in\mathbb{N}}$ of partial sums is increasing.

($\Rightarrow$) If $\sum_{r=1}^{\infty} a_r$ converges then $\{s_n\}_{n\in\mathbb{N}}$ converges by definition. Hence, by Theorem 3.2, $\{s_n\}_{n\in\mathbb{N}}$ is bounded.

($\Leftarrow$) Conversely, if $\{s_n\}_{n\in\mathbb{N}}$ is bounded then, in particular, it is bounded above. Since $\{s_n\}_{n\in\mathbb{N}}$ is also increasing, then $\{s_n\}_{n\in\mathbb{N}}$ is convergent by Theorem 3.4. Thus we have verified the definition that $\sum_{r=1}^{\infty} a_r$ is convergent.■

Remark If the series of non-negative terms $\sum_{r=1}^{\infty} a_r$ is convergent, the sequence $\{s_n\}_{n\in\mathbb{N}}$ is convergent and its limit, which is the sum of the series, is the $lub\{s_n : n \in \mathbb{N}\}$. (See Theorem 3.4.)

The next result is a way of testing convergence or divergence by comparison with a known series.

Theorem 5.2 (First Comparison Test)

Let $\sum_{r=1}^{\infty} a_r$ and $\sum_{r=1}^{\infty} b_r$ be series with $0 \leq a_r \leq b_r$ for all $r \in \mathbb{N}$.

(i) If $\sum_{r=1}^{\infty} b_r$ is convergent then $\sum_{r=1}^{\infty} a_r$ is convergent. If $\sum_{r=1}^{\infty} b_r$ has sum τ and $\sum_{r=1}^{\infty} a_r$ has a sum σ, then $\sigma \leq \tau$.

(ii) If $\sum_{r=1}^{\infty} a_r$ is divergent, then $\sum_{r=1}^{\infty} b_r$ is divergent.

Proof

(i) Let s_n and t_n be the n^{th} partial sums of $\sum_{r=1}^{\infty} a_r$ and $\sum_{r=1}^{\infty} b_r$, respectively. As in the proof of Theorem 5.1 both $\{s_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ are increasing sequences.

By hypothesis, $\{t_n\}_{n\in\mathbb{N}}$ is convergent with limit τ. But $\{t_n\}_{n\in\mathbb{N}}$ is increasing, so by Theorem 3.4, τ is the least upper bound of $\{t_n : n \in \mathbb{N}\}$.

Since $0 \leq a_r \leq b_r$ for all $r \in \mathbb{N}$, we have that

$$0 \leq \sum_{r=1}^{n} a_r \leq \sum_{r=1}^{n} b_r,$$

i.e. $0 \leq s_n \leq t_n$ for all $n \in \mathbb{N}$. Thus all the s_n are no greater than any upper bound of $\{t_n : n \in \mathbb{N}\}$, that is, $s_n \leq \tau$ for all $n \in \mathbb{N}$. So τ is **an** upper bound for $\{s_n : n \in \mathbb{N}\}$.

Then, since $\{s_n\}_{n\in\mathbb{N}}$ is also increasing, we have again by Theorem 3.4 that $\{s_n\}_{n\in\mathbb{N}}$ is convergent with limit $\sigma = lub\{s_n : n \in \mathbb{N}\}$. Being the **least** of all upper bounds σ is less than or equal to any upper bound of the $\{s_n : n \in \mathbb{N}\}$. In particular, $\sigma \leq \tau$.

(ii) Again, this is simply the contrapositive of part (i) (See the appendix within section 3 of these notes.) ■

Example Show that $\sum_{r=0}^{\infty} \frac{1}{3^r+1}$ is convergent and $\sum_{r=1}^{\infty} \frac{1}{r^{2/3}}$ is divergent.

Solution Firstly,

$$0 \leq \frac{1}{3^r + 1} \leq \frac{1}{3^r}$$

and $\sum_{r=0}^{\infty} \frac{1}{3^r}$ converges since it is a Geometric Series with ratio $\frac{1}{3}$ (See Theorem 4.1). Hence our series converges.

Secondly,

$$0 \leq \frac{1}{r} \leq \frac{1}{r^{2/3}}$$

and the fact that $\sum_{r=1}^{\infty} \frac{1}{r}$ diverges is an earlier example. Hence our series diverges. ■

See also Question 6 Sheet 5

Theorem 5.3 (Second Comparison Test)

Let $\sum_{r=1}^{\infty} a_r$ and $\sum_{r=1}^{\infty} b_r$ be series such that $a_r \geq 0$ and $b_r > 0$ for all $r \in \mathbb{N}$. Suppose that the sequence $\left\{\frac{a_n}{b_n}\right\}_{n\in\mathbb{N}}$ is convergent with limit $\ell \neq 0$.

Then $\sum_{r=1}^{\infty} a_r$ is convergent if and only if $\sum_{r=1}^{\infty} b_r$ is convergent.

Proof

Suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = \ell$. Since $a_n \geq 0$ and $b_n > 0$ we have $\frac{a_n}{b_n} \geq 0$ and thus $\ell \geq 0$. But, by assumption, $\ell \neq 0$, hence $\ell > 0$.

We now apply Lemma 3.6, concluding that there exists $N_0 \in \mathbb{N}$ such that

$$\frac{\ell}{2} < \frac{a_n}{b_n} < \frac{3\ell}{2} \tag{11}$$

for all $n \geq N_0$.

($\Rightarrow$) First suppose that $\sum_{r=1}^{\infty} a_r$ is convergent.

By Theorem 4.2 $\sum_{r=N_0}^{\infty} a_r$ is convergent.

By Theorem 4.4 $\sum_{r=N_0}^{\infty} \frac{2}{\ell} a_r$ is convergent.

From (11) we have

$$0 < b_n < \frac{2}{\ell} a_n$$

for all $n \geq N_0$. So, by the First Comparison Test, $\sum_{r=N_0}^{\infty} b_r$ is convergent. Finally, by Theorem 4.2 again, $\sum_{r=1}^{\infty} b_r$ is convergent.

($\Leftarrow$) Conversely, suppose that $\sum_{r=1}^{\infty} b_r$ is convergent.

By Theorem 4.2 $\sum_{r=N_0}^{\infty} b_r$ is convergent.

By Theorem 4.4 $\sum_{r=N_0}^{\infty} \frac{3\ell}{2} b_r$ is convergent.

This time we use (11) in the form

$$0 \leq a_n < \frac{3\ell}{2} b_n$$

for all $n \geq N_0$. So, by the First Comparison Test, $\sum_{r=N_0}^{\infty} \frac{2}{3\ell} a_r$ is convergent. Again$\sum_{r=1}^{\infty} a_r$ is convergent, justified by Theorems 4.2. ■

Note If the sequence $\{a_n/b_n\}_{n\in\mathbb{N}}$ is either divergent or has a zero limit then Theorem 5.3 tells us nothing. We have to either choose a different series $\sum b_r$ for comparison or use a different test on our given series $\sum a_r$.

We can use the Comparison tests to prove the following..

Theorem 5.4

$$\sum_{r=1}^{\infty} \frac{1}{r^2}$$

is convergent.

Solution. As before, the idea is to compare this series with

$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)}.$$

This may not look a "simpler" series but we saw in Theorem 4.8 that it is easy to sum.

Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n(n+1)}$. Then $\frac{a_n}{b_n} = 1 + \frac{1}{n}$ and so $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \neq 0$. Hence by the Second Comparison test, Theorem 5.3, $\sum_{r=1}^{\infty} \frac{1}{r^2}$ is convergent.■

Excercise for students; try to show that $\sum_{r=1}^{\infty} \frac{1}{r^2}$ converges, with sum less than 2, using the *First* Comparison Test.

Note In later courses it will be shown that $\sum_{r=1}^{\infty} \frac{1}{r^2}$ has sum $\pi^2/6$.

Theorem 5.4 For $k \in \mathbb{Z}$ we have that

$$\sum_{r=1}^{\infty} \frac{1}{r^k} \quad \text{is} \quad \begin{cases} \text{convergent if } k \geq 2 \\ \text{divergent if } k \leq 1. \end{cases}$$

Proof (Left to student)

Example Test the series

$$\sum_{r=1}^{\infty} \frac{2r^2 + 2r + 1}{r^5 + 2}$$

for convergence.

Solution

Rough work

For large r, $2r^2 + 2r + 1$ is dominated by $2r^2$ (i.e. if $r = 1,000$ then $2r^2$ differs from $2r^2 + 2r + 1$ by less than 0.1%). Similarly $r^5 + 2$ is dominated by r^5, so for large r the sum will "look like" $\sum_r \frac{2}{r^3}$ which we know, by Theorem 5.4, converges.

End of rough work

Let

$$a_n = \frac{2n^2 + 2n + 1}{n^5 + 2}, \quad \text{and} \quad b_n = \frac{1}{n^3}.$$

Then

$$\frac{a_n}{b_n} = \frac{n^3(2n^2 + 2n + 1)}{n^5 + 2} = \frac{2 + \frac{2}{n} + \frac{2}{n^2}}{1 + \frac{2}{n^5}}, \quad \text{so} \quad \lim_{n \to \infty} \frac{a_n}{b_n} = 2 \neq 0.$$

Since, by Theorem 5.4, $\sum_{r=1}^{\infty} \frac{1}{r^3}$ is convergent, we can use the Second Comparison Test to deduce that $\sum_{r=1}^{\infty} \frac{2r^2+2r+1}{r^5+2}$ converges. ■

Example Test the series

$$\sum_{r=1}^{\infty} \frac{r^2 - 2r - 3}{r^3 - 2}$$

for convergence.

Proof

Rough work

For large r the general term of this series will "look like" $\frac{r^2}{r^3} = \frac{1}{r}$, the sum of which we know diverges.

End of rough work

Let

$$a_n = \frac{n^2 - 2n - 3}{n^3 - 2}, \quad \text{and} \quad b_n = \frac{1}{n}.$$

Then

$$\frac{a_n}{b_n} = \frac{n(n^2 - 2n - 3)}{n^3 - 2} = \frac{1 - \frac{2}{n} - \frac{3}{n^2}}{1 - \frac{2}{n^3}}, \quad \text{so} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = 1 \neq 0.$$

Since by an example above, the Harmonic series $\sum_{r=1}^{\infty} \frac{1}{r}$ is divergent, we can use the Second Comparison Test to deduce that $\sum_{r=1}^{\infty} \frac{r^2-2r-3}{r^3-2}$ diverges. ■

Exercise for student: try to prove the last result using the *First* Comparison Test.

Remark In the last example we have cheated slightly as $a_r < 0$ when $r = 2$. The Comparison Test requires $a_r \geq 0$ for all r. However, this does not matter because we can apply the test to $\sum_{r=3}^{\infty} a_r$ and deduce that this is divergent. Then $\sum_{r=1}^{\infty} a_r$ must also be divergent. Thus the Comparison Tests can be applied to series $\sum_{r=1}^{\infty} a_r$ which have at most a finite number of negative terms.

Appendix

Theorem 5.5 For $k \in \mathbb{Z}$ we have that

$$\sum_{r=1}^{\infty} \frac{1}{r^k} \quad \text{is} \quad \begin{cases} \text{convergent if } k \geq 2 \\ \text{divergent if } k \leq 1. \end{cases}$$

Proof If $k \geq 2$ then

$$0 < \frac{1}{r^k} \leq \frac{1}{r^2}.$$

for all $r \in \mathbb{N}$. By Theorem 5.4, $\sum_{r=1}^{\infty} \frac{1}{r^2}$ is convergent. So by the First Comparison Test, Theorem 5.2, we deduce that $\sum_{r=1}^{\infty} \frac{1}{r^k}$ is convergent.

If $k \leq 1$ then

$$\frac{1}{r} \leq \frac{1}{r^k}$$

for all $r \in \mathbb{N}$. We have seen earlier that the Harmonic series, $\sum_{r=1}^{\infty} \frac{1}{r}$, is divergent. So by the First Comparison Test, Theorem 5.2, we deduce that $\sum_{r=1}^{\infty} \frac{1}{r^k}$ is divergent. ■

Note I have restricted to $k \in \mathbb{Z}$ in Theorem 5.5 since I have not defined r^k when $r \in \mathbb{N}$, for a general $k \in \mathbb{R}$. For example, how would we define $2^{\sqrt{2}}$ or 3^{π}?

But we can define r^k when $k \in \mathbb{Q}$. For when $k \in \mathbb{Q}$ we can write $k = p/q$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then we can define $r^k = \left(r^{1/q}\right)^p$ where $r^{1/q}$ is the positive real root of $x^q - r = 0$.

With this definition we can extend Theorem 5.5: Let $k \in \mathbb{Q}$. Then

$$\sum_{r=1}^{\infty} \frac{1}{r^k} \qquad \text{is} \begin{cases} \text{convergent if } k > 1 \\ \text{divergent if } k \leq 1. \end{cases}$$

This shows that the case $k = 1$, the Harmonic series, is on the boundary between convergence and divergence. In particular, it diverges but it does so slowly.