Section 4 Series

Definition Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. The infinite sum

$$a_1 + a_2 + a_3 + ...$$

is called a **series**. We call a_n the n-th term of the series. We denote $a_1 + a_2 + a_3 + ...$ by $\sum_{r=1}^{\infty} a_r$.

Examples

$\sum_{r=1}^{\infty}(-1)^r r = (-1) + 2 + (-3) + 4 + ... ,$
$\sum_{r=1}^{\infty} \frac{1}{r} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ... ,$
$\sum_{r=1}^{\infty} \frac{1}{r^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + ...$

Definition Let $a_1+a_2+a_3+...$ be a series. For each $n \in \mathbb{N}$, the n**-th partial sum** is the sum of the first n terms, i.e.

$$s_n = a_1 + a_2 + a_3 + ... + a_n \quad \text{or} \quad s_n = \sum_{r=1}^{n} a_r. \tag{5}$$

Examples $\sum_{r=1}^{\infty}(-1)^r r$. In this case the sequence of partial sums starts $s_1 = -1$, $s_2 = 1$, $s_3 = -2$, $s_4 = 2, ...$, that is we get the sequence $-1, 1, -2, 2, -3, 3, -4, 4, ...$.

$\sum_{r=1}^{\infty} \frac{1}{r}$. In this case we get the sequence $1, 1.5.1.8\overline{3}, 2.08\overline{3}, 2.28\overline{3}, 2.45, 2.5928, ..$
$\sum_{r=1}^{\infty} \frac{1}{r^2}$. In this case we get $1, 1.25, 1.36\overline{1}, 1.42361\overline{1}, 1.46361\overline{1}, 1.4913\overline{8}, ...$

So given a *series* $a_1 + a_2 + a_3 + ...$ we obtain a *sequence* of partial sums $s_1, s_2, s_3, ...$. This sequence is either convergent or divergent.

Definition The *series* $a_1+a_2+a_3+...$ is said to be **convergent** if the *sequence* of partial sums $\{s_n\}_{n\in\mathbb{N}}$ is convergent. In this case the limit, $\lim_{n\to\infty} s_n$, is called the **sum** of the series $a_1 + a_2 + a_3 + ...$.

The series $a_1 + a_2 + a_3 + ...$ is said to be **divergent** if the sequence of partial sums $\{s_n\}_{n\in\mathbb{N}}$ is divergent.

Example
Show that $\sum_{r=1}^{\infty}(-1)^r r$ is divergent,
Solution
As we saw above the partial sums are $-1, 1, -2, 2, -3, 3, -4, 4, ...$ which can be given by the formula $-\left(\frac{n+1}{2}\right)$ if n odd and $\frac{n}{2}$ if n even. So the sequence

is unbounded and thus it diverges. Hence the series diverges. ■

Question Do $\sum_{r=1}^{\infty} \frac{1}{r}$ and $\sum_{r=1}^{\infty} \frac{1}{r^2}$ converge or diverge? We cannot answer these questions until after Corollary 4.6 and Theorem 5.4 respectively.

Definition With $x, \lambda \in \mathbb{R}$, then $\sum_{r=0}^{\infty} \lambda x^r$ is called a **geometric series**, λ is the **first term** and x is the **common ratio** between successive terms. **Note** that this series starts at 0 and not 1, and we take x^0 to be 1 for **all** x, so the partial sum of the first n terms is

$$s_n = \lambda + \lambda x + \lambda x^2 + ... + \lambda x^{n-1}. \tag{6}$$

We can calculate this sum for general x.

Theorem 4.1 Let $\lambda, x \in \mathbb{R}$ with $\lambda \neq 0$.
(i) If $|x| < 1$, then $\sum_{r=0}^{\infty} \lambda x^r$ is convergent with sum $\frac{\lambda}{1-x}$.
(ii) If $|x| \geq 1$, then $\sum_{r=0}^{\infty} \lambda x^r$ is divergent.

Proof
Let s_n be the n^{th} partial sum so, for all x,

$$\begin{aligned} xs_n &= x(\lambda + \lambda x + \lambda x^2 + ... + \lambda x^{n-1}) \qquad \text{by (6)}, \\ &= \lambda x + \lambda x^2 + \lambda x^3 ... + \lambda x^n + \lambda x^n \\ &\qquad \text{by the distributive law, allowable since only a finite number of additions in bracket,} \\ &= (\lambda + \lambda x + \lambda x^2 + ... + \lambda x^{n-1}) - \lambda + \lambda x^n \qquad \text{``adding in zero''} \\ &= s_n - \lambda + \lambda x^n. \end{aligned}$$

Thus

$$(1 - x)s_n = \lambda(1 - x^n). \tag{7}$$

(i) If $|x| < 1$ then $1 - x \neq 0$ so we can rearrange (7) to get

$$s_n = \frac{\lambda(1 - x^n)}{(1 - x)}. \tag{8}$$

By Theorem 3.10 $\{x^n\}_{n\in\mathbb{N}}$ converges, with limit 0. Hence, by Corollary 3.8, $\{s_n\}_{n\in\mathbb{N}}$ converges with limit $\frac{\lambda}{1-x}$.

Thus $\sum_{r=0}^{\infty} \lambda x^r$ converges with sum $\dfrac{\lambda}{1-x}$.

(ii) If $x = 1$ then from (6) we have $s_n = \lambda n$ for all $n \geq 1$ and so, since $\lambda \neq 0$, the sequence of partial sums diverges.

If either $x = -1$ or $|x| > 1$ then $1 - x \neq 0$ and so we get (8) again. But Theorem 3.10 tells us this time that $\{x^n\}_{n\in\mathbb{N}}$ diverges as must $\{s_n\}_{n\in\mathbb{N}}$ (again using $\lambda \neq 0$).

So in all cases when $|x| \geq 1$ we have that $\{s_n\}_{n\in\mathbb{N}}$, and thus the geometric series, diverges. ■

As noted above, the geometric series starts at $r = 0$ while I chose the general series to start at $r = 1$ (so that the formula for s_n in (5) was straightforward). This makes no difference to convergence as is seen in

Theorem 4.2 Let $\sum_{r=0}^{\infty} a_r$ be a series and let $k \in \mathbb{N}$. Then $\sum_{r=0}^{\infty} a_r$ is convergent if and only if $\sum_{r=k}^{\infty} a_r$ is convergent. If $\sum_{r=0}^{\infty} a_r$ has sum σ then $\sum_{r=k}^{\infty} a_r$ has a sum $\sigma - (a_0 + ... + a_{k-1})$.

Proof (Left to student but see the appendix.)

Example Show that

$$\sum_{r=1}^{\infty} \frac{1}{2^r}$$

is convergent with sum 1.

Solution

Note that this series does not start at $r = 0$ as geometric series should. Instead we can let $j = r - 1$ in which case the sum over j will start at 0 and is given by

$$\sum_{j=0}^{\infty} \frac{1}{2^{j+1}} = \sum_{j=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^j$$

which is now of the correct form and so we can apply Theorem 4.1 (a) with $x = 1/2$ and $\lambda = 1/2$ to get

$$\sum_{j=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^j = \frac{1/2}{(1-1/2)} = 1.$$

Alternatively we can apply Theorem 4.2. So we first evaluate $\sum_{r=0}^{\infty} \frac{1}{2^r}$, which we do by applying Theorem 4.1(a) with $x = 1/2$ and $\lambda = 1$ to get

$$\sum_{r=0}^{\infty} \frac{1}{2^r} = \frac{1}{(1-1/2)} = 2.$$

Then Theorem 4.2 gives

$$\sum_{r=1}^{\infty} \frac{1}{2^r} = \sum_{r=0}^{\infty} \frac{1}{2^r} - 1 = 1.$$

■

It was easy to sum the geometric series in Theorem 4.1 because the partial sums s_n in (6) had a simple form in (8). There are other cases where the partial sums have a simple form.

Theorem 4.3 The series

$$\sum_{r=1}^{\infty} \frac{1}{r\,(r+1)}$$

converges with sum 1.

Proof

A simple application of partial fractions shows that

$$\frac{1}{r\,(r+1)} = \frac{1}{r} - \frac{1}{r+1}.$$

So the n^{th} partial sum can be written as

$$\begin{aligned} s_n &= \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + ... + \frac{1}{n.\,(n+1)} \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + ... + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \frac{1}{4}\right) + ... + \left(-\frac{1}{n} + \frac{1}{n}\right) - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

Hence the sequence $\{s_n\}_{n\in\mathbb{N}}$ converges as, thus, does the series with sum equal to $\lim_{n\to\infty} s_n = 1$. ■

See Question 4 sheet 4

The next question is what operations can we do to infinite sums. For instance, for *finite* sums we know that

$$\lambda(a_1 + a_2 + ... + a_n) = \lambda a_1 + \lambda a_2 + ... + \lambda a_n.$$

This can be proved by applying the distributive law (i.e. Property 3 of $\mathbb{R}$) $n-1$ times. But can we say

$$\lambda(a_1 + a_2 + ...) = \lambda a_1 + \lambda a_2 + ... \tag{9}$$

when we have infinite series. We don't have the "time" to apply the distributive law infinitely many times. But if we remember that the value we give to an infinite sum is a limit, $\lim_{n\to\infty} s_n$, we can recall a result on limits of convergent sequences, Corollary 3.8(i), that gives $\lambda \lim_{n\to\infty} s_n = \lim_{n\to\infty} \lambda s_n$. Then for each n we will be able to use the distributive law to say something of λs_n. In this way we can prove (9). Similarly, the associative law (Property 2 of $\mathbb{R}$) gives

$$\begin{aligned}&(a_1 + a_2 + ... + a_n) + (b_1 + b_2 + ... + b_n)\\ &\quad = (a_1 + b_1) + (a_2 + b_2) + ... + (a_n + b_n).\end{aligned}$$

By looking at limits we can give the infinite sum result

$$(a_1 + a_2 + ...) + (b_1 + b_2 + ...) = (a_1 + b_1) + (a_2 + b_2) + ...$$

or

$$\sum_{r=1}^{\infty} a_r + \sum_{r=1}^{\infty} b_r = \sum_{r=1}^{\infty} (a_r + b_r).$$

Both these results are given in the following.

Theorem 4.4 Let $\sum_{r=1}^{\infty} a_r, \sum_{r=1}^{\infty} b_r$ be convergent series with sums σ and τ respectively and let $\lambda, \mu \in \mathbb{R}$. Then the series $\sum_{r=1}^{\infty}(\lambda a_r + \mu b_r)$ is convergent with sum $\lambda\sigma + \mu\tau$.

Proof

Let s_n, t_n be the n^{th} partial sums of $\sum_{r=1}^{\infty} a_r, \sum_{r=1}^{\infty} b_r$ respectively. Then the n^{th} partial sum

$$\begin{aligned}\sum_{r=1}^{n}(\lambda a_r + \mu b_r) &= (\lambda a_1 + \mu b_1) + (\lambda a_2 + \mu b_2) + ... + (\lambda a_n + \mu b_n)\\ &= (\lambda a_1 + \lambda a_2 + ... + \lambda a_n) + (\mu b_1 + \mu b_2 ... + \mu b_n)\\ &\qquad \text{associative law}\\ &= \lambda(a_1 + a_2 + ... + a_n) + \mu(b_1 + b_2 + ... + b_n)\\ &\qquad \text{distributive law}\\ &= \lambda s_n + \mu t_n.\end{aligned}$$

But we are given that $\lim_{n\to\infty} s_n = \sigma$ and $\lim_{n\to\infty} t_n = \tau$ so by Theorem 3.7 and Corollary 3.8 we find that $\{\lambda s_n + \mu t_n\}_{n\in\mathbb{N}}$ is convergent with

$\lim_{n\to\infty}(\lambda s_n + \mu t_n) = \lambda\sigma + \mu\tau$. Therefore , by definition, $\sum_{r=1}^{\infty}(\lambda a_r + \mu b_r)$ is indeed convergent with sum $\lambda\sigma + \mu\tau$. ■

With this result we can combine convergent series to form new convergent series. Alternatively we can use the result to decompose complicated series into simpler ones. This often helps in checking whether a series is convergent.

Example Evaluate

$$2 - \frac{1}{3} + \frac{5}{9} - \frac{7}{27} + \frac{17}{81} - \frac{31}{243} + \ldots .$$

Solution We first have to find a formula for the n^{th}-term. We quickly see that each denominator is a power of 3, starting with $1 = 3^0$. Then there is an alternating sign, $(-1)^r$, if we start with $r = 0$. The numerators are more difficult, but the $5, 7, 17, 31, \ldots$ should remind one of $4, 8, 16, 32, \ldots$ i.e. powers of 2. In fact, $5 = 2^2 + 1$, $7 = 2^3 - 1$, $17 = 2^4 + 1$, $31 = 2^5 - 1$. In general, $2^r + (-1)^r$. Combine all together to see the sum is

$$\sum_{r=0}^{\infty} \frac{1 + (-2)^r}{3^r}.$$

We examine the series

$$\sum_{r=0}^{\infty} \frac{1}{3^r} \qquad \text{and} \qquad \sum_{r=0}^{\infty} \frac{(-2)^r}{3^r}.$$

If they are convergent *then* Theorem 4.4 tells us that our original series is convergent. But both of these simpler series are geometric series with $\lambda = 1$ in both cases and ratios $x = \frac{1}{3}, -\frac{2}{3}$ respectively. Since $|x| < 1$ in both cases the geometric series converge as does the original series. But Theorem 4.4 says, further, that we can add the sums of the simpler series together to get the sum of the original series. From Theorem 4.1 the sums are $1/(1 - (1/3)) = 3/2$ and $1/(1 - (-2/3)) = 3/5$ respectively. Hence

$$\sum_{r=0}^{\infty} \frac{1 + (-2)^r}{3^r} = \frac{3}{2} + \frac{3}{5} = \frac{21}{10}.$$

■

Theorem 4.5 Let $\sum_{r=1}^{\infty} a_r$ be a convergent series. Then the *sequence* $\{a_n\}_{n\in\mathbb{N}}$ is convergent with limit 0, that is, $\lim_{n\to\infty} a_n = 0$.

Proof

Let s_n be the n^{th} partial sum of $\sum_{r=1}^{\infty} a_r$ and let σ denote the sum of this series; so $\{s_n\}_{n\in\mathbb{N}}$ has limit σ.

Define a new sequence

$$t_n = \begin{cases} s_{n-1} & \text{if } n > 1 \\ 0 & \text{if } n = 1. \end{cases}$$

Then $\{t_n\}_{n\in\mathbb{N}}$ also has limit σ. Hence, by Corollary 3.8, $\{s_n - t_n\}_{n\in\mathbb{N}}$ is convergent with limit $\sigma - \sigma = 0$. But $s_n - t_n = a_n$, and thus $\lim_{n\to\infty} a_n = 0$. ■

Re-expressing Theorem 4.5 we obtain a test for *divergence.*

Corollary 4.6 Let $\sum_{r=0}^{\infty} a_r$ be such that the *sequence* $\{a_n\}_{n\in\mathbb{N}}$ is either divergent or convergent with a non-zero limit then the *series* $\sum_{r=0}^{\infty} a_r$ is divergent.

Proof This is simply the "contrapositive" of Theorem 4.5 where, to recall, the contrapositive of "If p then q" is "If not q then not p". The hardest part here is to see that the negation of "$\{a_n\}_{n\in\mathbb{N}}$ converges to zero" is "either $\{a_n\}_{n\in\mathbb{N}}$ is divergent or convergent with a non-zero limit". ■

For applications see Question 3 Sheet 5

Example Let $\lambda, x \in \mathbb{R}$ with $\lambda \neq 0$ and $|x| \geq 1$. Then the geometric series $\sum_{r=0}^{\infty} \lambda x^r$ diverges.

Solution The terms of the series satisfy $|\lambda x^r| \geq |\lambda|$, since $|x| \geq 1$, and then since $\lambda \neq 0$ we see that the terms of the sequence $\{\lambda x^r\}_{r\geq 1}$ cannot converge to 0. Thus by Corollary 4.6 the geometric series diverges. ■

This is an alternative proof to Theorem 4.1(ii).

Note, as discussed in the Appendix to part 3 of the web notes, the *converse* of "If p then q" is "If q then p". The converse of Theorem 4.5 states that if $\lim_{n\to\infty} a_n = 0$ then $\sum_{r=0}^{\infty} a_r$ converges.

THIS IS FALSE!

There exists sequences $\{a_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} a_n = 0$ but for which $\sum_{r=0}^{\infty} a_r$ diverges. We see this easily in

Example The series $\displaystyle\sum_{r=1}^{\infty} \frac{1}{\sqrt{r}}$ diverges even though $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Solution. The n^{th} partial sum satisfies

$$\begin{aligned} s_n &= \sum_{r=1}^{n} \frac{1}{\sqrt{r}} \\ &> n \times \frac{1}{\sqrt{n}} \end{aligned}$$

(numbers of terms $\times$ lower bound for the first n terms), i.e. $s_n > \sqrt{n}$. This means that the sequence of partial sums is unbounded and thus diverges. Hence $\sum_{r=1}^{\infty} \frac{1}{\sqrt{r}}$ diverges. ■

See also Question 4 Sheet 5

This example is very simple but a far more important example is given in

Theorem 4.7 The *Harmonic series,* $\sum_{r=1}^{\infty} \frac{1}{r}$, diverges.

Proof Rough work

The idea of the proof is

$$\begin{aligned} & 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + ... \\ = \; & 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + ... + \frac{1}{16}\right) + \\ & + \left(\frac{1}{17} + ... + \frac{1}{32}\right) + ... \\ > \; & 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + ... + \frac{1}{16}\right) + \\ & + \left(\frac{1}{32} + ... + \frac{1}{32}\right) + ... \\ = \; & 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ... \, . \end{aligned}$$

The final series obviously diverges.

End of rough work

Step 1 For all $k \geq 1$.

$$s_{2^k} \geq \frac{k+2}{2} \tag{10}$$

Proof of step 1 is by induction.

When $k = 1$ we find that

$$s_{2^1} = 1 + \frac{1}{2} = \frac{3}{2} = \frac{1+2}{2}$$

and so (10) holds with equality.

Assume (10) holds when $k = r$, so $s_{2^r} \geq (r+2)/2$. Consider

$$\begin{aligned} s_{2^{r+1}} &= s_{2^r} + \frac{1}{2^r+1} + \frac{1}{2^r+2} + \frac{1}{2^r+3} + ... + \frac{1}{2^{r+1}} \\ &\geq s_{2^r} + 2^r \times \frac{1}{2^{r+1}} \\ &\quad \text{bounding the sum by the number of terms } \times \text{ smallest term} \\ &\geq \frac{r+2}{2} + \frac{1}{2} \quad \text{by inductive hypothesis} \\ &= \frac{(r+1)+2}{2}. \end{aligned}$$

Hence result holds for $k = r + 1$.

Thus by induction (10) holds for all $k \geq 1$.

Step 2 Show that $\{s_n\}_{n\in\mathbb{N}}$ is unbounded.

Proof of step 2 by contradiction

Assume that $\{s_n\}_{n\in\mathbb{N}}$ is bounded, by λ, say, so $s_n \leq \lambda$ for all $n \geq 1$.

By the alternative Archimedean Property there exists $k \in \mathbb{N}$ such that $k > 2\lambda - 2$, i.e. $\frac{k+2}{2} > \lambda$.

Then, by step 1, $s_{2^k} \geq \frac{k+2}{2} > \lambda$, so λ is not an upper bound.

Contradiction, so our assumption is false, thus $\{s_n\}_{n\in\mathbb{N}}$ is unbounded.

Finally, since the sequence of partial sums s_n is unbounded it diverges and so, by definition, the series $\sum_{r=1}^{\infty} \frac{1}{r}$ also diverges. ■

Remember:

$\sum_{r=1}^{\infty} a_r$ convergent	$\Rightarrow$	$\lim_{n\to\infty} a_n = 0$,
	but	
$\lim_{n\to\infty} a_n = 0$	$\not\Rightarrow$	$\sum_{r=1}^{\infty} a_r$ convergent

Appendix

Theorem 4.2 Let $\sum_{r=0}^{\infty} a_r$ be a series and let $k \in \mathbb{N}$. Then $\sum_{r=0}^{\infty} a_r$ is convergent if and only if $\sum_{r=k}^{\infty} a_r$ is convergent. If $\sum_{r=0}^{\infty} a_r$ has sum σ then $\sum_{r=k}^{\infty} a_r$ has a sum $\sigma - (a_0 + ... + a_{k-1})$.

Proof (Not for examination.)

Let s_n be the n^{th} partial sum of $\sum_{r=0}^{\infty} a_r$ so $s_n = a_0 + a_1 + ... + a_{n-1}$ and let t_n the n^{th} partial sum of $\sum_{r=k}^{\infty} a_r$. Thus

$$\begin{aligned} t_n &= a_k + a_{k+1} + ... + a_{n+k-1} \\ &= (a_0 + ... + a_{n+k-1}) - (a_0 + a_1 + ... + a_{k-1}) \\ &= s_{n+k} - (a_0 + a_1 + ... + a_{k-1}). \end{aligned}$$

Now $\{s_n\}$ is convergent if, and only if $\{s_{n+k}\}$ converges which happens if, and only if, $s_{n+k} - (a_0 + ... + a_{k-1})$ converges (use Corollary 3.7 with $b_n = a_0 + ... + a_{k-1}$ for all n), i.e. if, and only if $\{t_n\}$ converges.

Also, the sum of $\sum_{r=k}^{\infty} a_r$ equals, by definition,

$$\begin{aligned} \lim_{n\to\infty} t_n &= \lim_{n\to\infty} (s_{n+k} - (a_0 + ... + a_{k-1})) \\ &= \lim_{n\to\infty} s_{n+k} - (a_0 + ... + a_{k-1}) \\ &= \lim_{n\to\infty} s_n - (a_0 + ... + a_{k-1}) \\ &= \sigma - (a_0 + ... + a_{k-1}). \end{aligned}$$

■