153 Additional Questions

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The majority of the questions below build on and develop further some of the questions given on the Question Sheets. There is no solution sheet though any numerical values that are asked for are listed at the end and (hopefully a sufficient number of) hints are given.

1) Let $\alpha, \beta \in \mathbb{A}$ be real roots of $x^2 - 2 = 0$ and $x^3 - 3 = 0$ respectively.

(i) Find an equation with rational coefficients satisfied by $\alpha + \beta$. Hence show that $\alpha + \beta \in \mathbb{A}$.

(ii) Prove that $\alpha\beta \in \mathbb{A}$.

Aside: It can be shown that if α and β are **any** elements of $\mathbb{A}$ then both $\alpha + \beta, \alpha\beta \in \mathbb{A}$. Further, if $\alpha \in \mathbb{A} \setminus \{0\}$ then $\alpha^{-1} \in \mathbb{A}$. So $\mathbb{A}$ forms a field, i.e. satisfies the same Properties 1-5 as does $\mathbb{R}$.

2) (a) Show that $\sqrt[3]{3\sqrt[2]{21} + 8}$ is irrational.

(b) What about $\sqrt[3]{3\sqrt[2]{21} - 8}$?

(c) On a calculator evaluate each of the following

$$\begin{array}{ll} \text{(i)} & \sqrt[3]{3\sqrt[2]{21}+8} - \sqrt[3]{3\sqrt[2]{21}-8}, \\ \text{(ii)} & \sqrt[3]{20-14\sqrt[2]{2}} + \sqrt[3]{20+14\sqrt[2]{2}}, \\ \text{(iii)} & \sqrt[3]{2\sqrt[2]{13}+5} - \sqrt[3]{2\sqrt[2]{13}-5}. \end{array}$$

In each case the expression can be simplified in the following way.

(1) Find the cubic equation that the expression satisfies. To this end it might help to first show, by substitution perhaps, that if x is of the form $\sqrt[3]{\alpha} - \sqrt[3]{\beta}$ then x satisfies

$$x^3 + 3\sqrt[3]{\alpha\beta}x - (\alpha - \beta) = 0.$$

(2) For each of the three cubic equations you have derived find a real, *integer*, root. (The calculation made in (c) should help here).

(3) Factor each of the three cubics into a product of a linear and a quadratic factor.

(4) Show that in each case the quadratic factor has no real roots. (For this you need to look at the discriminant. For a quadratic $ax^2 + bx + c$ the discriminant is $b^2 - 4ac$. Where have you seen this before?)

(5) Conclude that each of the original numbers must equal the only real root of the corresponding cubic, a root that is an integer!

(6) Try to construct some examples of your own, i.e. try to find conditions on a, b and $c \in \mathbb{N}$ such that

$$\sqrt[3]{a\sqrt[2]{b}+c} - \sqrt[3]{a\sqrt[2]{b}-c} \in \mathbb{N}.$$

(d) In the same way show that

$$\sqrt[3]{\sqrt[2]{5}+2} + \sqrt[3]{\sqrt[2]{5}-2} = \sqrt[2]{5}.$$

(e) Further, find simpler expressions for

$$\begin{aligned} &\text{(i)} \qquad \sqrt[4]{28+8\sqrt[2]{12}} - \sqrt[4]{28-8\sqrt[2]{12}}, \\ &\text{(ii)} \qquad \sqrt[4]{196+16\sqrt[2]{150}} - \sqrt[4]{196-16\sqrt[2]{150}}. \end{aligned}$$

To prove your result you should follow the above method, though this time you start by finding a quartic equation the number satisfies. To this end it might help to first show that if x is of the form $\sqrt[4]{\gamma} - \sqrt[4]{\delta}$ then x satisfies

$$x^4 + 4\sqrt[4]{\gamma\delta}x^2 - \left(\sqrt[2]{\gamma} - \sqrt[2]{\delta}\right)^2 = 0.$$

3) Find all zeros of $x^4 - 4x^2 + x + 2 = 0$.

Hint: Look first for the rational zeros by writing

$$x^4 - 4x^2 + x + 2 = (x-p)\left(x^3 + ax^2 + bx + c\right)$$

where $p \in \mathbb{Z}$. (So why must p divide 2?). When the rational zeros have been found use them to factorize $x^4 - 4x^2 + x + 2$. You should then be able to find the zeros of any non-linear factor.

4) Which of the Properties 1-5 are satisfied by

(i) the set of all 2×2 matrices with real entries,

(ii) the set of all 2×2 matrices with real entries and determinant 1,

(iii) the set of all subsets of $\mathbb{R}$, with $\cup$ in place of $+$ and $\cap$ in place of $\times$.

If a property fails to hold, give a counter-example.

5) Define $<_{\mathbb{C}}$ on $\mathbb{C}$ as follows: for all $z, w \in \mathbb{C}$ set $z <_{\mathbb{C}} w$ if either $|z| < |w|$ or we have both $|z| = |w|$ and $\arg z < \arg w$.

Which of the properties 6-9 hold for $<_{\mathbb{C}}$?

If a property fails to hold, give a counter-example.

6) Let $\alpha \in \mathbb{R}$. Is

$$\begin{aligned} lub\,\{x < \alpha : x \in \mathbb{R}\} &= lub\,\{x < \alpha : x \in \mathbb{Q}\}? \\ lub\,\{x < \alpha : x \in \mathbb{R}\} &= lub\,\{x < \alpha : x \in \mathbb{Z}\}? \\ lub\,\{x < \alpha : x \in \mathbb{Q}\} &= lub\,\{x < \alpha : x \in \mathbb{Z}\}? \end{aligned}$$

7) Is $\mathbb{A}$ complete?

Hint: Think of a number, γ, not in $\mathbb{A}$, (you may have to take it on trust that it is not in $\mathbb{A}$). Find an increasing sequence of rationals (which are thus in $\mathbb{A}$) converging to γ. So $r_1 < r_2 < r_3 < ...$ with $\lim_{n\to\infty} r_n = \gamma$. Consider $R = \cup_{n\geq 1} \{x \in \mathbb{A} : x < r_n\}$. What can you say of $lubR$?

8)(i) Recall

$$a - b = \left(\sqrt{a} - \sqrt{b}\right)\left(\sqrt{a} + \sqrt{b}\right) \qquad \text{is valid for all } a, b \geq 0.$$

Let $\{a_n\}_{n\in\mathbb{N}}$ be a convergent sequence of positive terms, so $a_n > 0$ for all $n \geq 1$, with limit $\ell > 0$.

Prove that

$$\left|\sqrt{a_n} - \sqrt{\ell}\right| < \frac{|a_n - \ell|}{\sqrt{\ell}}.$$

(ii) Deduce, using the Archimedean Principle, that

$$\lim_{n\to\infty} \sqrt{a_n} = \sqrt{\ell}.$$

Can you prove this result when $\ell = 0$?

9) Calculate the limit of the sequence

$$\left\{(-1)^n \sqrt{n + (-1)^n} + (-1)^{n+1} \sqrt{n + (-1)^{n+1}}\right\}_{n\in\mathbb{N}}$$

10) Evaluate

$$\lim_{n\to\infty} \frac{\sqrt{n+1} + \sqrt{n-1}}{\sqrt{n+2} + \sqrt{n+1}}.$$

11) (i) Let $a, b, c, d > 0$. What is the value of

$$\lim_{n\to\infty}\left(\sqrt{n+a}\sqrt{n+b}-\sqrt{n+c}\sqrt{n+d}\right)?$$

(ii) Use $(a^2-ab+b^2)(a+b)=a^3+b^3$ to prove

$$\frac{n}{(n+1)^{1/3}} \le (n-1)^{2/3}-\left(n^2-1\right)^{1/3}+(n+1)^{2/3} \le \frac{n}{(n-1)^{1/3}}$$

for all $n \ge 2$. Hence evaluate

$$\lim_{n\to\infty}\left(\frac{(n-1)^{2/3}-(n^2-1)^{1/3}+(n+1)^{2/3}}{n^{2/3}}\right).$$

12) *Note*: The inequality in Question 8(i) of Sheet 4 can be deduced from the binomial expansion. The binomial expansion states that for $n \in \mathbb{N}$ we have

$$(1+\delta)^n = \sum_{j=0}^{n}\binom{n}{j}\delta^j \qquad \text{where} \qquad \binom{n}{j}=\frac{n!}{j!\,(n-j)!}$$

is the binomial coefficient. If we assume that $\delta > 0$ then, when we truncate the series at $j = 2$, we throw away positive terms and get as a lower bound the inequality in Question 8(i) of Sheet 4. Importantly, in Bernoulli's Lemma we have $\delta > -1$, i.e. the possibility of *negative* δ.

i) Assume $k, n \in \mathbb{N}$ satisfy $1 \le k \le n$. Prove that

$$(1+\delta)^n > \frac{n!}{k!\,(n-k)!}\delta^k$$

for all $\delta > 0$.

ii) Justify

$$\frac{n!}{(n-k)!} \ge 2\left(\frac{n}{2}\right)^k$$

for $n \ge 2\,(k+1)$.

iii) Assume $|x| < 1$. Prove that, given any $\ell \in \mathbb{N}$,

$$\left|n^\ell x^n - 0\right| < \frac{2^\ell\,(\ell+1)!}{n\delta^{\ell+1}}$$

for $n \ge 2\,(\ell+2)$, where $1+\delta = 1/x$. Deduce that $\lim_{n\to\infty} n^\ell x^n = 0$.

13) Recall

$$(1+\delta)^n \geq 1 + n\delta + \frac{n\,(n-1)}{2}\delta^2 \tag{1}$$

holds for all $n \geq 1$.

(i) Let $n \geq 1$ be given. Write $n^{1/n} = 1+\delta$ so $n = (1+\delta)^n$. Apply (1) to show

$$n > \frac{n\,(n-1)}{2}\delta^2.$$

(ii) Deduce that

$$\left|n^{1/n} - 1\right| < \sqrt{\frac{2}{n-1}} \qquad \text{for } n \geq 2.$$

(iii) Conclude that

$$\lim_{n\to\infty} n^{1/n} = 1.$$

14) Justify the following steps: For $r \in \mathbb{N}$,

$$\left(1+\frac{1}{r^2}\right)^{-r} = \left(1 - \frac{1}{r^2+1}\right)^r \geq 1 - \frac{r}{r^2+1}.$$

Thus

$$1 < \left(1+\frac{1}{r^2}\right)^r \leq 1 + \frac{r}{r^2-r+1}.$$

Hence

$$\lim_{r\to 1}\left(1+\frac{1}{r^2}\right)^r = 1.$$

15) Assume that $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of non-zero terms, so $a_n \neq 0$ for all $n \in \mathbb{N}$. Further, assume that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lambda$$

exists. Prove

(i) If $\lambda < 1$ then $\{a_n\}_{n\in\mathbb{N}}$ converges with limit 0,

(ii) If $\lambda > 1$ then $\{a_n\}_{n\in\mathbb{N}}$ diverges.

Hence, for any fixed $k, \ell \in \mathbb{N}$, $\ell \geq 2$, show that

$$\lim_{n\to\infty} \frac{n^k}{\ell^n} = 0.$$

16) (i) What is the smallest $n_0 \in \mathbb{N}$ for which $2^n \leq (n-1)!$ holds?

(ii) Prove that

$$2^n \leq (n-1)! \quad \text{for all} \quad n \geq n_0.$$

(iii) Deduce

$$\lim_{n\to\infty} \frac{2^n}{n!} \cos\left(\frac{\pi n}{8}\right) = 0.$$

17) Define $\{b_n\}_{n\in\mathbb{N}}$ iteratively by $b_1 = 1$ and

$$b_{n+1} = 1 + \frac{1}{b_n + 1}$$

for all $n \geq 1$. This sequence was seen in Question 7(ii), Sheet 2.

(i) Prove $b_n \geq 1$ for all $n \geq 1$.

(Hint: importantly, $b_n \geq 1$ implies $b_n > 0$.)

(ii) Prove that $b_n \leq 3/2$ for all $n \geq 1$.

(iii) Prove that $\{b_n\}_{n\in\mathbb{N}}$ is an increasing sequence.

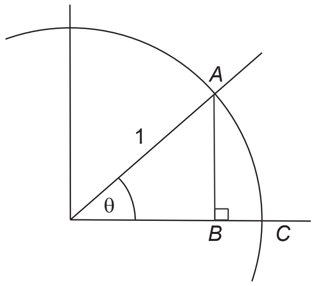
(iv) Deduce that $\lim_{n\to\infty} b_n$ exists and find its value.

18). Let $\{a_n\}_{n\in\mathbb{N}}$ be a convergent sequence with $\lim_{n\to\infty} a_n = 0$. Prove the following

(i)

$$\lim_{n\to\infty} \sin a_n = 0,$$

Hint: Calculate the lengths of the lines AB and AC in



and so show that for $0 < \theta < \pi/2$ we have $0 < \sin\theta < \theta$. A similar result follows for $\theta < 0$ since sin is a odd function.

(ii)

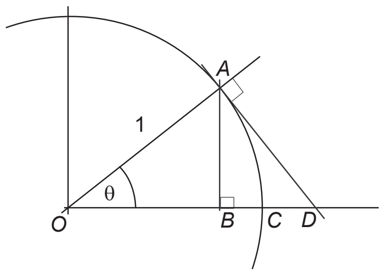
$$\lim_{n\to\infty} \cos a_n = 1,$$

(iii)

$$\lim_{n\to\infty} \frac{\sin a_n}{a_n} = 1$$

with the further condition that $a_n \neq 0$ for all $n \geq 1$.

Hint Calculate the lengths OB, AB and AD followed by the areas OAB, OAC and OAD in



This should give you $\theta\cos^2\theta < \sin\theta\cos\theta < \theta$ for $0 < \theta < \pi/2$.

Recall the double angle formula,

$$\cos\theta = 2\cos^2(\theta/2) - 1,$$

so

$$2\cos(\theta/2) = \sqrt{2 + 2\cos\theta},$$

where, if we start with $|\theta| < \pi/2$, we take the positive square root.

Define $a_n = 2\cos(\theta/2^{n-1})$. Show that $\{a_n\}_{n\geq 1}$ satisfies $a_{n+1} = \sqrt{2 + a_n}$ for all $n \geq 1$, a definition seen in Question 10 on Sheet 3 and Question 7(v) on Sheet 2. Show that $\lim_{n\to\infty} a_n = 2$ *whatever* the value of $\theta : |\theta| < \pi/2$ (i.e. whatever the value of $a_1 : |a_1| \leq 2$).

Prove that

$$\lim_{n\to\infty} 2^{n-1}\sqrt{(2 - a_n)} = \theta.$$

Hint:

$$2 - a_n = \frac{2^2 - a_n^2}{2 + a_n} = \frac{4(1 - \cos^2(\theta/2^{n-1}))}{2 + a_n} = \frac{4\sin^2(\theta/2^{n-1})}{2 + a_n},$$

and then use part (iii) above.

This is a proof of the result you might have guessed in question 7(vi) on sheet 2 (where $\theta = \pi/4$).

19) (a) Use the result (†) in Question 2 on Sheet 3 to solve the equation

$$\sqrt{2x^2 + 3x + 5} + \sqrt{2x^2 - 3x + 5} = 3x. \qquad (2)$$

Hint: The equation here is of the form

$$\sqrt{f(x)} + \sqrt{g(x)} = \phi(x) \qquad (3)$$

Multiplying both sides by $\sqrt{f(x)} - \sqrt{g(x)}$ gives

$$f(x) - g(x) = \phi(x)\left(\sqrt{f(x)} - \sqrt{g(x)}\right). \qquad (4)$$

Thus we have two equations $(3), (4)$ and, if we temporarily look upon $\sqrt{f(x)}$ and $\sqrt{g(x)}$ as the two unknowns, we can solve for $\sqrt{f(x)}$ and $\sqrt{g(x)}$. The answers would be of the form $\sqrt{f(x)} = k(x)$ and $\sqrt{g(x)} = \ell(x)$. These are both "simpler" equations than the original in that there is only one square root in each of these equations unlike the two in (2). It should be straightforward to solve these simpler equations. It would need to be checked that any solutions of these easier equations are, in fact, solutions of the original.

20)(a) Verify that

$$x^m - y^m = (x - y)\left(x^{m-1} + yx^{m-2} + y^2x^{m-3} + ... + y^{m-2}x + y^{m-1}\right). \qquad (5)$$

(i) If $0 < y < x$ show that

$$m(x-y)y^{m-1} \leq x^m - y^m \leq m(x-y)x^{m-1}.$$

Deduce that

$$\frac{j^{m+1} - (j-1)^{m+1}}{m+1} \leq j^m \leq \frac{(j+1)^{m+1} - j^{m+1}}{m+1}.$$

Hence prove that

$$\frac{n^{m+1}}{m+1} \leq \sum_{j=1}^{n} j^m \leq \frac{(n+1)^{m+1}-1}{m+1} = \frac{n^{m+1}}{m+1} + n^m + \dots .$$

So we might expect that $\sum_{j=1}^{n} j^m$ will be a polynomial in n of degree $m+1$ and with leading coefficient $\frac{1}{m+1}$. Check this with the known results for $m = 1$ and 2 (results usually proved using induction). Perhaps you can find the results for $m = 3$ and 4.

(b) Write

$$S(m) = \sum_{j=1}^{n} j^m.$$

Start from the binomial expansion in the form

$$(j+1)^{m+1} - j^{m+1} = \sum_{a=0}^{m} \binom{m+1}{a} j^a,$$

and sum over $1 \leq j \leq n$ to deduce

$$S(m) = \frac{(n+1)^{m+1}-1}{m+1} - \frac{1}{(m+1)} \sum_{a=0}^{m-1} \binom{m+1}{a} S(a).$$

This iterative result allows us to prove facts about $S(m)$ by induction.

Use it to verify the guess made in part (i) that the leading term of $S(m)$ is $n^{m+1}/(m+1)$.

What is the coefficient of n^m? of n^{m-1}?

21) Let $m \in \mathbb{N}$. Assume $\{a_n\}_{n\in\mathbb{N}}$ is a convergent sequence of positive terms with $\lim_{n\to\infty} a_n = \ell > 0$. Use (5) to show that

$$\left|a_n^{1/m} - \ell^{1/m}\right| < \frac{|a_n - \ell|}{m\ell^{(m-1)/m}}.$$

Prove that $\lim_{n\to\infty} a_n^{1/m} = \ell^{1/m}$.

Hint: Apply (5) with $(x, y) = \left(a_n^{1/m}, \ell^{1/m}\right)$.

Can you prove that $\lim_{n\to\infty} a_n^{k/m} = \ell^{k/m}$ for all $k, m \in \mathbb{N}$, that is, $\lim_{n\to\infty} a_n^r = \ell^r$ for all $r \in \mathbb{Q}, r > 0$?

22) Bernoulli's inequality states that

$$(1+\delta)^n > 1 + n\delta$$

for all $\delta > -1$ and all $n \in \mathbb{N}$. This result actually holds for all $n \in \mathbb{Q}$, $n \geq 1$. For $n = 3/2$ we can show this easily, as follows.

(i) Verify

$$(1+\delta)^{1/2} < 1 + \frac{\delta}{2} \tag{6}$$

for all $\delta > -1$. (Why not show that $(1+\delta)^{1/q} < 1 + \delta/q$ for $q \in \mathbb{N}$)

(ii) Deduce that

$$(1+\delta)^{3/2} = \frac{(1+\delta)^2}{(1+\delta)^{1/2}} > 1 + \frac{3\delta}{2} \tag{7}$$

for all $\delta > -1$. (Why not show that $(1+\delta)^{2-1/q} > 1 + (2-1/q)\delta$ for $q \in \mathbb{N}$.)

(iii) For an application of (7) consider any set of non-negative reals $a_i \geq 0$, $1 \leq i \leq m$ such that $a_1 + a_2 + ... + a_m = m$. Apply part (ii) to $(1 + (a_i - 1))^{3/2}$ for each i and add the results together to get

$$a_1^{3/2} + a_2^{3/2} + ... + a_m^{3/2} \geq m. \tag{8}$$

(iv) Further, take any set of m non-negative reals b_i, $1 \leq i \leq m$. Show that

$$\left(\frac{b_1^3 + b_2^3 + ... + b_m^3}{m}\right)^{1/3} \geq \left(\frac{b_1^2 + b_2^2 + ... + b_m^2}{m}\right)^{1/2}. \tag{9}$$

Hint: When the right hand side of (9) is non-zero apply (8) with $a_i = (b_i/r)^2$ where $r = (b_1^2 + b_2^2 + ... + b_m^2)/m$.

(v) Prove that

$$x + \frac{1}{x} \geq 2$$

for all $x > 0$. (It is not necessary to use calculus. Perhaps examine $(x-1)^2$.)

This can be rewritten as follows. If $a_1 a_2 = 1$ with $a_i > 0$, $1 \leq i \leq 2$, then

$$a_1 + a_2 \geq 2. \tag{10}$$

(vi) Extend this to three non-negative reals with $a_1a_2a_3 = 1$. First show that at least one of the a_i is ≤ 1. Then show that at least one of the a_i is ≥ 1. Then prove that $a_1 + a_2 + a_3 \geq 3$.

Hint: By relabelling if necessary assume that $a_2 \leq 1$ and $a_3 \geq 1$. Let $b_1 = a_1$ and $b_2 = a_2a_3$, so $b_1b_2 = 1$, and apply (10). Then use $a_2 \leq 1$ and $a_3 \geq 1$ to prove that $a_2 + a_3 \geq a_2a_3 + 1$. Combine to get result.

(vii) Prove by induction that if we have n non-negative reals satisfying $a_1a_2....a_n = 1$ then

$$a_1 + a_2 + ... + a_n \geq n. \tag{11}$$

(So though some of the a_i might be small the conditions $a_1a_2....a_n = 1$ means that others have to be large, so large in fact that (11) holds.)

Hint: As in (vi) justify the existence of an $a_i \leq 1$ and $a_j \geq 1$. Combine a_ia_j as one variable so reducing the number of terms and allowing the use of an inductive assumption. Finish in the same way as in (vi).

(viii) Extend this to the case when we are given any n non-negative reals, $b_i, 1 \leq i \leq n$, (so no condition on $b_1b_2...b_n$) and prove that

$$\frac{b_1 + b_2 + ... + b_n}{n} \geq (b_1b_2....b_n)^{1/n}. \tag{12}$$

We denote by

$$\mathcal{A}_n(\mathbf{b}) = \frac{b_1 + b_2 + ... + b_n}{n},$$

the *Arithmetic Mean* of the numbers $b_i, 1 \leq i \leq n$, and by

$$\mathcal{G}_n(\mathbf{b}) = (b_1b_2....b_n)^{1/n},$$

the *Geometric Mean* of the numbers. So (12) is known as the **AM-GM inequality**.

Hint for proof: When $\mathcal{G}_n(\mathbf{b}) \neq 0$ apply (11) with $a_i = b_i/\mathcal{G}_n(\mathbf{b})$ for all $1 \leq i \leq n$.

(ix) As an application of the AM-GM inequality let $y > -1$ and $1 \leq m \leq n$. Apply (12) with

$$\begin{aligned} b_1 &= b_2 = ... = b_{n-m} = 1 \\ b_{n-m+1} &= ... = b_n = 1 + y. \end{aligned}$$

Deduce that

$$\left(1+\frac{m}{n}y\right)^{n/m} \geq 1+y.$$

By an appropriate substitution prove that

$$(1+\delta)^{n/m} \geq 1+\frac{n}{m}\delta \tag{13}$$

for all $\delta > -1$. In this way we extend Bernoulli's inequality to all rational exponents, n/m, greater than 1.

From (13) deduce

$$(1+\delta)^{m/n} \leq 1+\frac{m}{n}\delta$$

for all $\delta > -1$. In this way we extend (6) to all positive rational exponents less than 1.

(x) As an application of (13) assume that $a_i, 1 \leq i \leq t$ are non-negative real numbers which satisfy $a_1 + a_2 + ... + a_t = t$. Then for any $1 \leq m \leq n$ show that

$$a_1^{n/m} + a_2^{n/m} + ... + a_t^{n/m} \geq t.$$

(xi) And further, if $b_i, 1 \leq i \leq t$ are non-negative then, for any $1 \leq m \leq n$,

$$\left(\frac{b_1^n + b_2^n + ... + b_t^n}{t}\right)^{1/n} \geq \left(\frac{b_1^m + b_2^m + ... + b_t^m}{t}\right)^{1/m}. \tag{14}$$

So (9) is a special case, $(m = 2, n = 3)$, of (14). We write

$$\mathcal{A}_t^n(\mathbf{b}) = \left(\frac{b_1^n + b_2^n + ... + b_t^n}{t}\right)^{1/n} \quad \text{for } n \geq 1, \tag{15}$$

in which case $\mathcal{A}_t^1(\mathbf{b}) = \mathcal{A}_t(\mathbf{b})$, the arithmetic mean defined above. So (14) can be written as $\mathcal{A}_t^n(\mathbf{b}) \geq \mathcal{A}_t^m(\mathbf{b})$.

In fact the definition (15) makes sense for all $-n \in \mathbb{N}$ as long as no $b_i = 0$. When $n = -1$ we get the *Harmonic Mean*

$$\mathcal{H}_t(\mathbf{b}) = \frac{t}{\frac{1}{b_1} + \frac{1}{b_2} + ... + \frac{1}{b_t}}.$$

(xii) Show that $\mathcal{H}_t(\mathbf{b}) \leq \mathcal{G}_t(\mathbf{b})$.

(xiii) Show that if $m < 0 < n$ then $\mathcal{A}_t^m(\mathbf{b}) < \mathcal{G}_t(\mathbf{b}) < \mathcal{A}_t^n(\mathbf{b})$

23) Calculate the n^{th}-partial sum of

$$\sum_{r=1}^{\infty} \frac{2r+1}{r^2\left(r+1\right)^2}.$$

What is the sum of this series?

24) Let $j \geq 1$. What is the value of the sum

$$\sum_{r=1}^{\infty} \frac{1}{r\left(r+1\right)...\left(r+j\right)}?$$

25) i) Write

$$R_j(x,n) = \sum_{r=1}^{n} r^j x^r.$$

So $R_1(x,n)$ was the subject of Question 9, Sheet 4. Show that

$$\begin{aligned} R_2(x,n) &= \sum_{r=0}^{n-1} \left(r+1\right)^2 x^r \\ &= xR_2(x,n) + 2xR_1(x,n) + xR_0(x,n) + x - \left(n+1\right)^2 x^{n+1}. \end{aligned}$$

Hence show that the value of the sum $\sum_{r=1}^{\infty} r^2 x^r$ is, for $|x| < 1$,

$$\frac{x^2+x}{\left(1-x\right)^3}.$$

ii) Start from the binomial expansion in the form

$$\left(r+1\right)^j - r^j = \sum_{a=0}^{j-1} \binom{j}{a} r^a$$

and sum over $1 \leq r \leq n$ to get

$$\left(1-x\right) R_j(x,n) = x\sum_{a=0}^{j-1} \binom{j}{a} R_a(x,n) + x - \left(n+1\right)^j x^{n+1}. \qquad (16)$$

(What result do you get when $x = 1$?)

Use (16) to show that **if** $\lim_{n\to\infty} R_a\left(x,n\right)$ exists for all $0 \leq a \leq j-1$ **then** $\lim_{n\to\infty} R_j\left(x,n\right)$ exists. Hence show that $\lim_{n\to\infty} R_j\left(x,n\right)$ exists for all $|x| < 1$ and $j \geq 1$.

Write $R_j(x) = \lim_{n\to\infty} R_j(x,n)$ for $|x| < 1$.

From looking at the first few values of j we might guess that

$$R_j(x) = \frac{xP_j(x)}{(1-x)^{j+1}},$$

where $P_j(x)$ is a polynomial with integer coefficients. Prove this is the case by giving a recursive formula for $P_j(x)$.

Further show that degree$P_j(x) = j-1$ and $P_j(0) = 1$ for all $j \geq 1$.

26) Find values for a, b and c such that

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + ... + \frac{n}{2^n} = a + \frac{b+cn}{2^n}$$

holds for all $n \in \mathbb{N}$.

Hint: The three cases $n = 1, 2$ or 3 will give three equations. You have three unknowns, a, b and c which you can then find. *Then* show that the stated equality holds for **all** $n \in \mathbb{N}$.

Deduce that

$$\sum_{r=1}^{\infty} r\left(\frac{1}{2}\right)^r$$

converges with sum 2.

Compare with Question 9 Sheet 4.

27) Let

$$s_k = 2 + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{k!}.$$

(i) Relate s_k to the corresponding k-th partial sum of $\sum_{r=1}^{\infty} 1/r(r+1)$. Hence show that $\sum_{r=1}^{\infty} 1/r!$ converges with sum less than 3.

(ii) As in Question 10 on Sheet 4 define

$$c_k = \left(1 + \frac{1}{k}\right)^k$$

for $k \geq 1$. Justify

$$c_k = \sum_{j=0}^{k} \left(1 - \frac{1}{k}\right)\left(1 - \frac{2}{k}\right)...\left(1 - \frac{j-1}{k}\right)\frac{1}{j!}. \qquad (17)$$

Deduce that $c_k < s_k$ for all $k \geq 1$.

(iii) Let $\{a_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers satisfying $0 < a_k < 1$ for all k. Prove by induction that

$$(1 - a_1)(1 - a_2)\dots(1 - a_n) \geq 1 - (a_1 + a_2 + \dots + a_n) \qquad (18)$$

for all $n \geq 1$.

(If $a_1 = a_2 = a_3 = \dots = a_n = \delta$ we get a special case of Bernoulli's inequality.)

Deduce that

$$c_{k^3} \geq \left(1 - \frac{1}{k}\right) s_{k-1}$$

for all $k \geq 1$.

Hint: In the expansion (17) of c_{k^3} throw away all terms with $j \geq k$. For the remaining terms use the inequality (18).

(iv) Conclude that

$$\sum_{j=0}^{\infty} \frac{1}{j!} = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k.$$

Hint: You may need to use the following facts, well-known from our course,

(a) A subsequence (say $\{c_{k^3}\}_{k\in\mathbb{N}}$) of a convergent sequence (i.e. $\{c_k\}_{k\in\mathbb{N}}$) converges to the same limit as the sequence,

(b) The product of two convergent sequences has as its limit the product of the limits of each individual sequences.

28) Though we do not cover integration in this course, we will assume in this question, that you know how to integrate $1/t$.

(i) Justify

$$\int_r^{r+1} \frac{dt}{t} \leq \frac{1}{r} \leq \int_{r-1}^{r} \frac{dt}{t}.$$

(ii) If $s_n = \sum_{r=1}^{n} 1/r$, justify

$$\frac{1}{n} + \ln n \leq s_n \leq 1 + \ln n$$

for all $n \geq 1$.

Aside: (a) In the notes we proved that $s_{2^k} \geq (k+1)/2$. The results here show that $s_{2^k} \geq (\ln 2)\, k = 0.69315..k$.

(b) It can be shown that there exists a constant, $\gamma = 0.577215663....$, Euler's constant, such that

$$\left|\sum_{r=1}^{n}\frac{1}{r}-\ln n-\gamma\right|\leq\frac{1}{n}$$

for all n sufficiently large.

29) (i) Let $\sum_{r=1}^{\infty} a_r$ be a series with n^{th}-partial sum s_n. Assume that $\lim_{n\to\infty} s_{2n} = \sigma$ and $\lim_{n\to\infty} a_n = 0$. Prove that $\lim_{n\to\infty} s_n = \sigma$. That is, as long as the terms of the series tend to zero we need only show that the sequence of even partial sums converges to be able to deduce that the series converges.

(ii) Use part (i) to show that

$$1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-...$$

converges.

30) By using the simple observation that $r^2 < r(r+1)$ prove that, in the notation of Question 1 on Sheet 5,

$$\frac{1}{m+1}<\sum_{r=0}^{\infty}\frac{1}{r^2}-s_m.$$

By using the results of Question 3(ii) on Sheet 4 show that the value of the sum of $\sum_{r=0}^{\infty}\frac{1}{r^2}$ lies in the interval $(1.6406, 1.6498)$.

Compare with Question 1 Sheet 5.

31) Prove that

$$r^3 > (r+1)\, r\, (r-1)$$

for all $r \geq 1$. Use this to prove

$$\sum_{r=0}^{\infty}\frac{1}{r^3}-s_m<\frac{1}{2(m+1)\,m}.$$

Give a lower bound of

$$\frac{1}{2(m+1)(m+2)}<\sum_{r=0}^{\infty}\frac{1}{r^3}-s_m.$$

By using the results of Question 3(iii) on Sheet 4 show that the value of the sum of $\sum_{r=0}^{\infty} \frac{1}{r^3}$ lies in the interval $(1.2013, 1.2021)$.

32) Use the Alternating Series test to show that the following series converge.

$$\text{(i)} \sum_{r=2}^{\infty} (-1)^{r-1} \frac{r^{99}}{(r-1)^{100}}, \qquad \text{(ii)} \sum_{r=1}^{\infty} (-1)^{r+1} \frac{r-1}{(r+2)^2}.$$

33) (i) Prove that

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{(2r-1)(2r)}$$

converges.

(ii) Write out the first eight or so terms of this series, using partial fractions on each summand.

34) (i) Write out enough terms of the sequence $\left\{(-1)^{n(n+1)/2}\right\}_{n\geq 1}$ for you to be convinced of a pattern.

(ii) Why can you write

$$\sum_{\substack{r=1 \\ r \text{ odd}}}^{\infty} \frac{(-1)^{(r+1)/2}}{r} \qquad \text{as} \qquad \sum_{j=1}^{\infty} \frac{(-1)^j}{2j-1}?$$

Use the Alternating Series Test to prove that this series converges.

(iii) Similarly, prove that

$$\sum_{\substack{r=1 \\ r \text{ even}}}^{\infty} \frac{(-1)^{r/2}}{r}$$

converges.

(iv) Combine (ii) and (iii) to deduce that

$$\sum_{r=1}^{\infty} \frac{(-1)^{r(r+1)/2}}{r}$$

converges.

35) Prove that $\sqrt{r+1} + \sqrt{r-1} < 2\sqrt{r}$ for all $r \geq 1$. Deduce that

$$\sqrt{r+1} - \sqrt{r-1} \geq \frac{1}{\sqrt{r}}.$$

Use this for $r \geq 2$ to prove that

$$\sqrt{n+1} + \sqrt{n} - \sqrt{2} > s_n$$

for all $n \geq 1$ where $s_n = \sum_{r=1}^{n} 1/\sqrt{r}$.

Is this in fact an improvement on the upper bound given in Question 4 on Sheet 5?

Find $N \in \mathbb{N}$ such that $N + 1 > s_{1000} > N$.

Could you have found this N using the bounds in Question 4?

36) Note that in Question 4 Sheet 5 if $s_n = \sum_{r=1}^{n} 1/\sqrt{r}$ then

$$1 \leq 2\sqrt{n} - s_n \leq 2\sqrt{2} - 1 - \frac{1}{\sqrt{n+1}}.$$

(i) Let $t_n = 2\sqrt{n} - s_n$. Prove that $\{t_n\}_{n\in\mathbb{N}}$ is an increasing sequence.

(ii) Deduce that

$$\lim_{n\to\infty} \left(2\sqrt{n} - s_n\right) = C_0$$

for some constant C_0 satisfying $1 < C_0 \leq 2\sqrt{2} - 1$.

Aside. We often write this result as

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r}} = 2\sqrt{n} + C_0 + E(n)$$

where $E(n)$ is an "error" term that satisfies $\lim_{n\to\infty} E(n) = 0$. We would do further study to see just how "fast" this error tends to zero. Why must we have $E(n) \geq 1/2\sqrt{n}$?

37) Assume $a_n > 0$ for all $n \geq 1$.

(i) Prove that if $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} 1/a_n$ diverges.
(ii) Give examples of sequences $\{a_n\}_{n\in\mathbb{N}}$ for which the following hold.

(α) $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges.

(β) $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ coverges.

38) (i) Justify the following steps.

Let $\ell \in \mathbb{N}, \ell \geq 2$ be given. Set $n_0 = \ell^4 + 2$. Then

$$\begin{aligned}(n_0 - 2)! &= \ell^4! = \prod_{j=1}^{\ell^4} j \geq \prod_{j=\ell^2}^{\ell^4} j \\ &\geq \prod_{j=\ell^2}^{\ell^4} \ell^2 > \ell^{2(\ell^4-\ell^2)} \geq \ell^{\ell^4+2} = \ell^{n_0}.\end{aligned}$$

(ii) Prove that $\ell^n \leq (n-2)!$ for all $n \geq n_0$.

(iii) Use the First Comparison Test to show that

$$\sum_{r=1}^{\infty} \frac{\ell^r}{r!}$$

converges.

39) (i) What is the smallest value, $r_0 \in \mathbb{N}$, for which $r_0! \leq r_0^{r_0-2}$?

Prove, by induction, that $r! \leq r^{r-2}$ for all $r \geq r_0$.

(ii) Prove that

$$\sum_{r=1}^{\infty} \frac{r!}{r^r}$$

converges

Compare with Question 12 Sheet 6.

40) We trivially have $r! \leq r^r$ for all $r \geq 1$. Here we try to improve this.

(i) Starting with $(n-2j)^2 \geq 0$, or otherwise, show that

$$(n-j)\, j \leq \frac{n^2}{4} \qquad \text{for all } 0 \leq j \leq n.$$

(ii) Use (i) to prove that

$$r! \leq \left(\frac{r+1}{2}\right)^r \qquad \text{for all } r \geq 1.$$

Hint: Look at $(r!)^2 = (r\,(r-1)\,(r-2)\,...2.1)^2$ and rearrange as

$$\begin{aligned} & [r \times 1]\,[(r-1)\times 2]\,[(r-2)\times 3]\,...\,[2\times (r-1)]\,[1\times r] \\ = & \prod_{j=1}^{r} [(r+1-j)\times j] \end{aligned}$$

and use (i) on each square bracket.

(iii) Can you find another rearrangement that gives

$$r! \leq 2\left(\frac{r}{2}\right)^r \qquad \text{for all } r \geq 1?$$

(iv) Use (ii) or (iii) to prove that

$$\sum_{r=1}^{\infty} \frac{r^r}{2^r r!} \text{ diverges.}$$

41): Improve Question 28 for $s_n = \sum_{j=1}^{n} 1/j$

(i)From

$$1 + \frac{1}{n} + \sum_{j=2}^{n-1} \frac{1}{j} = s_n = 1 + \frac{1}{2} + \sum_{j=3}^{n} \frac{1}{j}$$

deduce

$$\frac{1}{n} + C + \ln n \leq s_n \leq \frac{1}{2} + C + \ln n$$

for $n \geq 3$, where $0 \leq C = 1 - \ln 2 < 1/2$.

(ii) Prove that

$$\sum_{j=1}^{n} s_j = (n+1)\, s_n - n.$$

(iii) Combine (i) and (ii) and show

$$\sum_{j=1}^{n} \ln j \leq \ln 2 + \sum_{j=3}^{n} \left(s_j - C - \frac{1}{j} \right) \leq n \ln n - \frac{1}{2} n + C.$$

(iv) Prove

$$n! \leq n^n e^{-n/2} e^C = \frac{1}{2} n^n e^{-(n-2)/2},$$

for $n \geq 3$.

(v) Take any $k \in \mathbb{N}$. Show that there exists a constant C_k such that

$$\frac{1}{n} + C_k + \ln n \leq s_n \leq \frac{1}{k} + C_k + \ln n$$

for all $n \geq k$.

(vi) Use (v) in

$$\sum_{j=1}^{n} \ln j = \sum_{j=1}^{k-1} \ln j + \sum_{j=k}^{n} \ln j$$

to prove

$$\begin{array}{llll} (A) & \displaystyle\sum_{j=1}^{n} \ln j & \leq & n \ln n - \left(1 - \dfrac{1}{k}\right) n + D_k \\ (B) & \displaystyle\sum_{j=1}^{n} \ln j & \geq & (n+1) \ln n - \left(1 + \dfrac{1}{k}\right) n + E_k \end{array}$$

for some constants D_k and E_k, and valid for all $n \geq k$. So, in this range of n, we have

$$F_k n^{(n+1)} e^{-(1+1/k)n} \leq n! \leq G_k n^n e^{-(1-1/k)n}. \tag{19}$$

with $F_k = e^{E_k}$ and $G_k = e^{D_k}$.

Aside: It can be shown that

$$\lim_{n \to \infty} \frac{n! e^n}{n^n \sqrt{n}}$$

exists.

(vii) Use (19) with $k = 4$ to show that

$$\sum_{r=1}^{\infty} \frac{2^r r!}{r^r}$$

converges.

(viii) Use (19) with $k = 11$ to show that

$$\sum_{r=1}^{\infty} \frac{3^r r!}{r^r}$$

diverges.

Compare with Question 12 Sheet 6

42) Find the radius of convergence of

$$\sum_{r=1}^{\infty} \frac{r!}{r^r} x^r.$$

You will need to make use of Question 10 Sheet 4.

How does Question 41 relate to your answer?

43) Use the Alternating Series test to show that the following series converge.

$$\text{(i)} \sum_{r=2}^{\infty} (-1)^{r-1} \frac{r^{99}}{(r-1)^{100}}, \qquad \text{(ii)} \sum_{r=1}^{\infty} (-1)^{r+1} \frac{r-1}{(r+2)^2}.$$

44) (i) Give an example of a convergent series $\sum_{r=1}^{\infty} a_r$ for which $\sum_{r=1}^{\infty} a_r^2$ diverges.

(ii) Give an example of a convergent series $\sum_{r=1}^{\infty} a_r$ and a convergent sequence $\{b_n\}_{n\geq 1}$ with $\lim_{n\to\infty} b_n = 0$ for which $\sum_{r=1}^{\infty} a_r b_r$ diverges.

(iii) Give an example of a convergent series $\sum_{r=1}^{\infty} a_r$ for which $\sum_{r=1}^{\infty} (-1)^r a_r$ diverges.

45) (i) Can you use partial fractions to prove that

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)(2r)}$$

converges? Give your reasons.

Use an appropriate Comparison Test to prove that this series converges

(ii) Use (i) to prove that

$$\sum_{\substack{r=1 \\ r \text{ odd}}}^{\infty} \frac{1}{r(r+1)}$$

converges.

Show the partial sums of this series are closely connected with those of

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - .... .$$

Hence show that is latter series converges without using the Alternating series test.

(iii) Use (ii) to show that

$$\sum_{\substack{r=1 \\ r \text{ odd}}}^{\infty} \frac{(-1)^{r+1}}{r(r+1)}$$

converges.

Deduce that

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - ....$$

converges.

Could you have proved this using the Alternating Sign Test?

46) (i) Write out enough terms of the sequence $\left\{(-1)^{n(n+1)/2}\right\}_{n\geq 1}$ for you to be convinced of the pattern.

(ii) Why can you write

$$\sum_{\substack{r=1 \\ r \text{ odd}}}^{\infty} \frac{(-1)^{(r+1)/2}}{r} \qquad \text{as} \qquad \sum_{j=1}^{\infty} \frac{(-1)^{j}}{2j-1}?$$

Use the Alternating Sign Test to prove that this series converges.

(iii) Similarly, prove that

$$\sum_{\substack{r=1 \\ r \text{ even}}}^{\infty} \frac{(-1)^{r/2}}{r}$$

converges.

(iv) Combine (ii) and (iii) to deduce that

$$\sum_{r=1}^{\infty} \frac{(-1)^{r(r+1)/2}}{r}$$

converges.

47) Find the radius of convergence for

$$\text{(i)} \ \sum_{r=1}^{\infty} \left(1 + \frac{1}{r^2}\right)^{r} x^r \ \text{ and (ii)} \ \ \sum_{r=1}^{\infty} \left(1 + \frac{1}{r^2}\right)^{r^2} x^r.$$

48) Let $0 \leq \theta < 1$.
Define

$$s_n = \sum_{r=1}^{n} \frac{1}{r^\theta}.$$

Show that $s_n \geq n^{1-\theta}$ for all $n \geq 1$.

Hence show that

$$\sum_{r=1}^{\infty} \frac{1}{r^\theta}$$

diverges.

49) Let $\theta > 1$. Let

$$s_n = \sum_{r=1}^{n} \frac{1}{r^\theta}.$$

(i) Use the idea that a finite sum is $\leq$ largest term $\times$ number of terms to prove

$$s_{2^n-1} - s_{2^{n-1}-1} \leq \left(\frac{1}{2^{\theta-1}}\right)^{n-1}$$

for all $n \geq 1$. Deduce

$$s_{2^k-1} \leq \sum_{j=0}^{k-1} \left(\frac{1}{2^{\theta-1}}\right)^{j}$$

for all $k \geq 1$.

(ii) Prove that

$$s_{2^k-1} \leq \frac{2^\theta}{2^\theta - 1}$$

for all $k \geq 1$.

(iii) Prove that

$$\sum_{r=1}^{\infty} \frac{1}{r^\theta}$$

converges.

50) Let $\{a_n\}_{n\in\mathbb{N}}$ be a decreasing sequence converging to 0.

(i) Prove that

$$\sum_{r=1}^{\infty}(a_r - a_{r+1})$$

converges. What is it's sum?

(ii) Prove that

$$\sum_{r=1}^{\infty}(-1)^{r+1}(a_r - a_{r+1})$$

converges.

Hint: Does it converge absolutely?

(iii) Write

$$t_n = \sum_{r=1}^{n}(-1)^{r+1}(a_r - a_{r+1}) \text{ and } s_n = \sum_{r=1}^{n}(-1)^{r+1}a_r.$$

Prove that

$$t_n = 2s_n - a_1 + (-1)^n a_{n+1}.$$

(iv) Deduce that

$$\sum_{r=1}^{\infty}(-1)^{r+1}a_r$$

converges.

So we have a proof of the Alternating Sign Test.

Answers

2)(c) (i) 1, (ii) 4, (iii) 1,

2)(e) (i) 2, (ii) 4.

3) $-2, 1, \frac{(1+\sqrt{5})}{2}, \frac{(1-\sqrt{5})}{2}$.

9) 0,

10) 1,

11) (i) $(a+b)-(c+d)$,

(ii) 1,

17) (iv) $\sqrt{2}$.

19) $x = 4$.

20)(b) $n^m/2$, $mn^{m-1}/12$ and the constant term is zero for all $m \geq 1$.

23) $s_n = 1 - 1/\left(n+1\right)^2$.

24) The value of the sum is $\frac{1}{j.j!}$

25) (c) $P_j\left(x\right) = x\sum_{a=1}^{j-1}\binom{j}{a}\left(1-x\right)^{j-a-1}P_a\left(x\right) + \left(1-x\right)^{j-1}$ having used $P_0\left(x\right) \equiv 1$

35) $N = 61$.

40) (iii) $\left(r!\right)^2 = r^2\prod_{j=1}^{r-1}\left(r-j\right)j$.