153 Problem Sheet 6

All questions should be attempted. Those marked with a ** must be handed in for marking by your supervisor. Hopefully the supervisor will have time to cover at least the questions marked with a * or **. Questions marked with a # will be discussed in the problems class. Those marked with H are slightly harder than the others.

1**) Use the Alternating Series Test to show that the following series converge.

$$\text{(i)} \sum_{r=2}^{\infty} (-1)^{r-1} \frac{r}{(r-1)^2}, \qquad \text{(ii)} \sum_{r=1}^{\infty} (-1)^{r+1} \frac{r-1}{(r+2)^2}.$$

2*) Prove that the following series are convergent by proving they are absolutely convergent.

$$\text{(i)} \sum_{r=1}^{\infty} (-1)^{\frac{1}{2}r(r+1)} \frac{r+1}{r^3+1}, \qquad \text{(ii)} \sum_{r=1}^{\infty} \frac{r+1}{(-2)^r r^2}, \qquad \text{(iii)} \sum_{r=1}^{\infty} (-1)^r \frac{x^{2r+1}}{2r+1},$$

where $-1 < x < 1$ in part (iii).

3#) Show that the following series are conditionally convergent.

$$\text{(i)} \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{\sqrt{r}} \qquad \text{(ii)} \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{b_r} \qquad \text{where } b_n = n + \frac{1}{2} + \frac{3}{2}(-1)^{n+1}.$$

Hint: In part (ii) write out the first few terms of the series and evaluate the first few partial sums. There should be a lot of cancellation in these partial sums.

4#) Determine whether the following series converge or diverge.

$$\text{(i)} \sum_{r=1}^{\infty} \frac{\cos(\pi r)}{r}, \qquad \text{(ii)} \sum_{r=1}^{\infty} \frac{(-1)^r \cos(\pi r)}{r},$$

$$\text{(iii)} \sum_{r=1}^{\infty} \frac{(-1)^r \cos^2(\pi r)}{r}, \quad \text{(iv)} \sum_{r=1}^{\infty} \left(\frac{1 + r\cos(\pi r)}{r^2} \right).$$

5#) (i) Write down the first few values of

$$\begin{aligned} \sin\left(\frac{\pi}{4}+n\frac{\pi}{2}\right), \qquad n &= 0,1,2,3,4,5,... \\ \cos\left(\frac{\pi}{4}+n\frac{\pi}{2}\right), \qquad n &= 0,1,2,3,4,5,... . \end{aligned}$$

(ii) Use (i) to derive a simple expression for

$$\tan\left(\frac{\pi}{4}+n\frac{\pi}{2}\right), \text{ for all } n \geq 0.$$

Can you give a proof for your result?

(iii) Use (ii) to prove that

$$\sum_{r=1}^{\infty} \tan\left(\frac{\pi}{4}+r\frac{\pi}{2}\right)\frac{1}{r}$$

is conditionally convergent.

6*) (i) Give an example of a convergent series $\sum_{r=1}^{\infty} a_r$ for which $\sum_{r=1}^{\infty} a_r^2$ diverges.

(ii) Give an example of a convergent series $\sum_{r=1}^{\infty} a_r$ and a convergent sequence $\{b_n\}_{n\geq 1}$ with $\lim_{n\to\infty} b_n = 0$ for which $\sum_{r=1}^{\infty} a_r b_r$ diverges.

(iii) Give an example of a convergent series $\sum_{r=1}^{\infty} a_r$ for which $\sum_{r=1}^{\infty} (-1)^{r+1} a_r$ diverges.

7) (i) Can you use partial fractions to prove that

$$\sum_{r=1}^{\infty} \frac{1}{(2r-1)(2r)}$$

converges? Give your reasons.

Use an appropriate Comparison Test to prove that this series converges

8**) Use the Ratio Test to determine whether the following series are convergent or divergent.

$$\text{(i)} \sum_{r=1}^{\infty} \frac{r^3}{2^r}, \qquad \text{(ii)} \sum_{r=1}^{\infty} \frac{(r!)^2}{(2r)!},$$

$$\text{(iii)} \sum_{r=1}^{\infty} \frac{(3r)!}{(r!)^3}, \qquad \text{(iv)} \sum_{r=1}^{\infty} \left(\frac{r}{r+1}\right) \frac{2^{5(r+1)}}{5^{2(r-1)}}.$$

9*) In the following series, of the form $\sum_{r=1}^{\infty} a_r$, show that $a_{n+1} \geq a_n$ for all sufficiently large n. Hence deduce that these series diverge.

$$\text{(i)} \sum_{r=1}^{\infty} \frac{r!}{2^r}, \qquad \text{(ii)} \sum_{r=1}^{\infty} \frac{(2r)!}{6^r r!}.$$

Why can you **not** apply the Ratio Test to show these diverge?

10#) **If** you can use the ratio test to show that both $\sum_{r=1}^{\infty} a_r$ and $\sum_{r=1}^{\infty} b_r$ converge what can you say of $\sum_{r=1}^{\infty} a_r b_r$ and why?

11H) Is it possible to apply the Ratio Test to the series

$$\frac{1}{2^2} + \frac{1}{2^1} + \frac{1}{2^4} + \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^5} + ...?$$

Does this series converge?

12#) (i) From Question 10 on Sheet 4 we know that

$$\lim_{r \to \infty} \left(1 + \frac{1}{r}\right)^r = c$$

for some constant $2 < c < 3$. Use this to show that

$$\lim_{r \to \infty} \left(\frac{r}{r+1}\right)^r = c^{-1}.$$

(ii) Use the Ratio Test and (i) to see if

$$\sum_{r=1}^{\infty} \frac{r!}{r^r}$$

converges or diverges.

(iii) What can you say of

$$\text{(a)} \sum_{r=1}^{\infty} \frac{2^r r!}{r^r} \quad \text{and} \quad \text{(b)} \sum_{r=1}^{\infty} \frac{3^r r!}{r^r} \ ?$$

13) (i) Show that $r! \leq r^r$ for all $r \geq 1$.

(ii) Deduce that

$$\sum_{r=1}^{\infty} \frac{r^r}{r!}$$

diverges.

14#) (i) Prove that the series $\sum_{r=1}^{\infty} \dfrac{r}{r^2+1} x^r$ is convergent when $|x| < 1$ and divergent when $|x| > 1$. What happens when $x = 1$ or $x = -1$?

(ii) Determine **all** values of x for which the series $\sum_{r=1}^{\infty} \dfrac{(-1)^r}{r x^r}$ converges.

15**) Determine the radius of convergence for the following power series.

$$\text{(i)} \sum_{r=1}^{\infty} \frac{r^3 x^r}{r!}, \qquad \text{(ii)} \sum_{r=1}^{\infty} \frac{(r!)^2 x^r}{(2r)!}, \qquad \text{(iii)} \sum_{r=1}^{\infty} (3^r + 4^r)\, x^r.$$

16#) (i) Give an example of a divergent series $\sum_{r=1}^{\infty} a_r$ for which $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = 1$.

(ii) Give an example of a convergent series $\sum_{r=1}^{\infty} a_r$ for which $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = 1$.

(iii) Show that the Harmonic series is an example of a divergent series $\sum_{r=1}^{\infty} a_r$ for which $\lim_{n\to\infty} |a_n|^{1/n} = 1$.
Hint: Try to make use of Question 4 on the Additional Question Sheet .

(iv) Give an example of a convergent series $\sum_{r=1}^{\infty} a_r$ for which $\lim_{n\to\infty} |a_n|^{1/n} = 1$.

17*) Use Cauchy's n-th root test to determine whether the following converge.

$$\text{(i)} \sum_{r=1}^{\infty} \frac{1}{r^r}, \qquad \text{(ii)} \sum_{r=1}^{\infty} \left(2 + \frac{4}{r}\right)^r.$$

18) Determine **all** values of x for which the following series converge.

$$\text{(i)} \sum_{r=1}^{\infty} \left(\frac{r+1}{rx}\right)^r, \quad \text{(ii)} \sum_{r=1}^{\infty} \left(5 + \frac{4}{r}\right)^2 x^r, \text{(iii)} \sum_{r=1}^{\infty} \left(5 + \frac{4}{r}\right)^r x^r$$