153 Problem Sheet 5

All questions should be attempted. Those marked with a ** must be handed in for marking by your supervisor. Hopefully the supervisor will have time to cover at least the questions marked with a * or **. Questions marked with a # will be discussed in the problems class. Those marked with H are slightly harder than the others.

1#) Let

$$s_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} \qquad \text{for } n \in \mathbb{N}.$$

(i) Show that $\{s_n\}_{n\in\mathbb{N}}$ is an increasing sequence.

(ii) Justify the following bound,

$$\begin{aligned} s_n &< 1 + \frac{1}{2^2} + \frac{1}{3.2} + \frac{1}{4.3} + ... + \frac{1}{n(n-1)} \\ &= \frac{7}{4} - \frac{1}{n}. \end{aligned}$$

Hence, by verifying the conditions of Theorem 3.4, deduce that

$$\sum_{r=1}^{\infty} \frac{1}{r^2} \qquad (\dagger)$$

is convergent with sum no greater than $\frac{7}{4}$.

(The value of the limit is $\frac{\pi^2}{6} = 1.64493...$, but this is hard to show.)

2) Let $\sum_{r=1}^{\infty} a_r$ be a series of non-negative terms which is convergent with sum σ. Let $\{b_n\}_{n\in\mathbb{N}}$ be a subsequence of $\{a_n\}_{n\in\mathbb{N}}$.

Prove that $\sum_{r=1}^{\infty} b_r$ is convergent with its sum τ satisfying $\tau \le \sigma$.

(Hint: Show that σ is an upper bound for $\{t_n : n \in \mathbb{N}\}$, where t_n is the n-th partial sum for the series $\sum_{r=1}^{\infty} b_r$.)

3*) Prove that the following series are divergent.

$$\text{(i)} \ \sum_{r=1}^{\infty} \frac{r}{r+1} \qquad\qquad \text{(ii)} \ \sum_{r=1}^{\infty} (-1)^r \left(r - \sqrt{r(r-1)} \right).$$

(Hint: Apply Corollary 4.6 and, for part (ii), use Question 2 on Sheet 3.)

4#) Let

$$s_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}}.$$

Recall that we saw in the notes that $s_n > \sqrt{n}$ for all $n \geq 1$.
(i) Prove that for all $r \in \mathbb{N}$ we have

$$2\left(\sqrt{r} - \sqrt{r-1}\right) \geq \frac{1}{\sqrt{r}} \geq 2\left(\sqrt{r+1} - \sqrt{r}\right).$$

(Hint: Use (†) from Question 2 on Sheet 3 twice.)
(ii) Deduce

$$2\sqrt{n} - 1 \geq s_n \geq 2\sqrt{n+1} + 1 - 2\sqrt{2}$$

for all $n \geq 1$ and further show

$$2\sqrt{n} - 1 \geq s_n \geq 2\sqrt{n} + 1 - 2\sqrt{2} + \frac{1}{\sqrt{n+1}}.$$

5)(i) Let $\{a_n\}_{n \in \mathbb{N}}$ be a convergent sequence with limit ℓ. Prove that

$$\lim_{n \to \infty} (a_{n+1} - a_n) = 0.$$

(Hint: Look at $|a_{n+1} - \ell + \ell - a_n|$).
(ii) Give an example of an increasing sequence $\{a_n\}_{n \geq 1}$ that is **not** bounded above (and so diverges) but for which

$$\lim_{n \to \infty} (a_{n+1} - a_n) = 0.$$

(Hint: Think about using the Harmonic series in some way.)

6#) Use the First Comparison Test to determine whether the following series are convergent or divergent.

$$\text{(i)} \sum_{r=0}^{\infty} \frac{1}{2^r + 3^r}, \qquad \text{(ii)} \sum_{r=1}^{\infty} \frac{1}{r3^r}, \qquad \text{(iii)} \sum_{r=1}^{\infty} \frac{1}{r^{4/5}},$$

$$\text{(iv)} \sum_{r=2}^{\infty} \frac{1}{\sqrt{r^2 - 1}}, \qquad \text{(v)} \sum_{r=1}^{\infty} \frac{1 + \sin r}{3r^2 + r}.$$

7**) Use the Second Comparison Test to determine whether the following series are convergent or divergent.

$$\text{(i)} \sum_{r=1}^{\infty} \frac{r+1}{r^3 + 2}, \qquad \text{(ii)} \sum_{r=1}^{\infty} \sqrt{\frac{r-1}{r^3}}, \qquad \text{(iii)} \sum_{r=1}^{\infty} \frac{2^r}{3^r - 1}.$$

8*) Use the Alternating Series Test to prove that the following series are convergent.

$$\text{(i)} \sum_{r=1}^{\infty} (-1)^{r+1} \frac{1}{2r}, \qquad \text{(ii)} \sum_{r=1}^{\infty} (-1)^{r+1} \left(\sqrt{r+1} - \sqrt{r-1} \right).$$

(Hint: In part (ii) look back at Question 2a(ii) on Sheet 3.)