153 Problem Sheet 3

All questions should be attempted. Those marked with a ** must be handed in for marking by your supervisor. Hopefully the supervisor will have time to cover at least the questions marked with a * or **. Questions marked with a # will be discussed in the problems class. Those marked with H are slightly harder than the others.

1)(i) Prove by induction that $2^n \geq n^2$ for all $n \geq 4$.

Deduce, using the Archimedean Principle, that

$$\lim_{n\to\infty} \frac{n}{2^n} = 0.$$

(ii) Prove by induction that $n^{n-1} \geq n!$ for all $n \geq 1$.

Deduce, using the Archimedean Principle, that

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

2#) The factorization of a difference of squares, namely $x^2-y^2 = (x-y)(x+y)$ valid for all $x, y \in \mathbb{R}$, can be used in the form

$$a - b = \left(\sqrt{a} - \sqrt{b}\right)\left(\sqrt{a} + \sqrt{b}\right) \qquad \text{valid for all } a, b \geq 0. \qquad (\dagger)$$

In this question we give three applications of ($\dagger$).

a)(i) Use ($\dagger$) with $(a, b) = (n+1, n-1)$ to prove that

$$\frac{1}{\sqrt{n+1}} < \sqrt{n+1} - \sqrt{n-1} < \frac{1}{\sqrt{n-1}}$$

holds for all $n \geq 2$.

a)(ii) Deduce, using the Archimedean Principle, that

$$\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n-1}\right) = 0.$$

(Hint: look back at Question 9(iv) on Sheet 2.)

We first looked at this sequence in Question 8 of sheet 2.

b)(i) Use ($\dagger$) with $(a, b) = \left(\frac{n}{n-1}, 1\right)$ to prove that

$$\left|\sqrt{\frac{n}{n-1}} - 1\right| < \frac{1}{2(n-1)}$$

holds for all $n \geq 2$.

b)(ii) Deduce, using the Archimedean Principle, that

$$\lim_{n\to\infty} \sqrt{\frac{n}{n-1}} = 1.$$

c)(i) Use (†) with $(a, b) = (n^2, n\,(n-1))$ to prove that

$$\frac{1}{2} \leq n - \sqrt{n(n-1)} \leq \frac{1}{2}\sqrt{\frac{n}{n-1}}$$

holds for all $n \geq 2$.

c)(ii) Deduce, using the Archimedean Principle, that

$$\lim_{n\to\infty} \left(n - \sqrt{n(n-1)}\right) = \frac{1}{2}.$$

(Hint: Use part b(ii) of this question.)

3*) For each of the following sequences, decide whether it is bounded, monotonic or convergent.

(i) $\left\{\frac{n-1}{n}\right\}_{n\in\mathbb{N}}$ (ii) $\left\{(-1)^n + \frac{1}{n}\right\}_{n\in\mathbb{N}}$ (iii) $\left\{\frac{n^2+1}{n}\right\}_{n\in\mathbb{N}}$ (iv) $\left\{1 - \frac{(-1)^n}{n}\right\}_{n\in\mathbb{N}}$.

4) Let $\{b_n\}_{n\in\mathbb{N}}$ be a subsequence of $\{a_n\}_{n\in\mathbb{N}}$. State, with reasons, whether the following are true or false.

(i) If $\{a_n\}_{n\in\mathbb{N}}$ has limit ℓ then $\{b_n\}_{n\in\mathbb{N}}$ has limit ℓ,

(ii) If $\{b_n\}_{n\in\mathbb{N}}$ is convergent then $\{a_n\}_{n\in\mathbb{N}}$ is convergent,

(iii) If $\{b_n\}_{n\in\mathbb{N}}$ is obtained by omitting a finite number of terms from $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ has limit ℓ then $\{a_n\}_{n\in\mathbb{N}}$ has limit ℓ.

(iv) If $\{a_n\}_{n\in\mathbb{N}}$ is convergent and $\{b_n\}_{n\in\mathbb{N}}$ has limit ℓ then $\{a_n\}_{n\in\mathbb{N}}$ has limit ℓ.

5) Suppose that $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are convergent sequences.

(i) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, prove that $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$;

(ii) Find examples of $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ with $a_n < b_n$ for all $n \in \mathbb{N}$ but for which $\lim_{n\to\infty} a_n \not< \lim_{n\to\infty} b_n$.

6) Using Corollary 3.8(ii), prove that if $\{a_n\}_{n\in\mathbb{N}}$ is convergent and $\{b_n\}_{n\in\mathbb{N}}$ is divergent, then $\{a_n + b_n\}_{n\in\mathbb{N}}$ is divergent.

7#) Find examples of sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ such that

(i) $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are divergent but $\{a_n+b_n\}_{n\in\mathbb{N}}$ is convergent;

(ii) $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are divergent but $\{a_nb_n\}_{n\in\mathbb{N}}$ is convergent;

(iii) $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are unbounded but $\{a_n+b_n\}_{n\in\mathbb{N}}$ is bounded yet divergent.

8**) Use Theorems 3.7 and 3.10 along with Corollary 3.8 to prove that the following sequences are convergent and find their limits.

(i) $\left\{\dfrac{n^2-n}{2n^2+1}\right\}_{n\in\mathbb{N}}$, (ii) $\left\{\dfrac{2n^2-3n+2}{n^3+1}\right\}_{n\in\mathbb{N}}$,

(iii) $\left\{\dfrac{\frac{1}{2}-\left(\frac{1}{3}\right)^n}{\frac{1}{3}-\left(\frac{1}{4}\right)^n}\right\}_{n\in\mathbb{N}}$ (iv) $\left\{\dfrac{4^n-3^n}{4^n-3}\right\}_{n\in\mathbb{N}}$,

(v) $\left\{\dfrac{2^n+n}{2^n-n}\right\}_{n\in\mathbb{N}}$, (vii) $\left\{\dfrac{5n!-6n^n}{6n!-5n^n}\right\}_{n\in\mathbb{N}}$.

9) (Exam 1997) Suppose that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is defined recursively by

$$a_1 = 1 \text{ and } \quad 2a_{n+1} = a_n + 3.$$

Prove that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is increasing and bounded above by 3. Hence, or otherwise, determine the limit $\lim_{n\to\infty} a_n$.

We first looked at this sequence in Question 7(i) on Sheet 2.

10#) Suppose that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is defined recursively by

$$a_1 = \sqrt{2} \text{ and } \quad a_{n+1} = \sqrt{a_n+2}.$$

Prove that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is increasing and bounded above by 2. Hence, or otherwise, determine $\lim_{n\to\infty} a_n$.

We first looked at this sequence in Question 7(v) on Sheet 2.