153 Problem Sheet 2

All questions should be attempted. Those marked with a ** must be handed in for marking by your supervisor. Hopefully the supervisor will have time to cover at least the questions marked with a * or **. Questions marked with a # will be discussed in the problems class. Those marked with H are slightly harder than the others.

1**) Prove, using Theorem 2.3, that 1 is the *glb* of the set $\left\{ \dfrac{n^2+1}{n^2} : n \in \mathbb{N} \right\}$.

2) In the notes we have defined the closed-open interval $[a,b) = \{x : a \leq x < b\}$ for $a, b \in \mathbb{R}$.

(i) Verify the definition of *glb* to prove that $a = glb\left([a,b)\right)$.

(ii) Use the definition of *lub* along with proof by contradiction to show that $b = lub\left([a,b)\right)$.

(iii) Alternatively, use Theorem 2.2 to show that $b = lub\left([a,b)\right)$.

3*) Let A and B be subsets of $\mathbb{R}$ that are bounded above. Show that $A \cup B$ is bounded above. What is $lub(A \cup B)$ in terms of $lubA$ and $lubB$?

4H) Let A_1 and A_2 be non-empty subsets of $\mathbb{R}$ which are bounded above with *lub*s β_1 and β_2 respectively. Define

$$A_1 + A_2 = \{a_1 + a_2 : a_1 \in A_1, a_2 \in A_2\}.$$

Show that $A_1 + A_2$ is bounded above with *lub* equal to $\beta_1 + \beta_2$.

(Hint: Show that $\beta_1 + \beta_2$ is the *lub* by verifying the conditions of Theorem 2.2.

So, first show that $\beta_1 + \beta_2$ is **an** upper bound.

Then show that, given any $\varepsilon > 0$, that $\beta_1 + \beta_2 - \varepsilon$ is not an upper bound by finding $a_1 \in A_1$ such that $a_1 > \beta_1 - \frac{\varepsilon}{2}$ and $a_2 \in A_2$ such that $a_2 > \beta_2 - \frac{\varepsilon}{2}$.

A different proof is given in the solution sheets.)

5#) Write down the formulae for a_n, the n-th term, of the following sequences.

(i) $1, 2, 4, 8, 16, \ldots,$

(ii) $0, 1, 0, 1, 0, 1, \ldots,$

(iii) $\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots,$

(iv) $-1, \frac{1}{3}, -\frac{1}{5}, \frac{1}{7}, -\frac{1}{9}, \frac{1}{11}, \ldots,$

(v) $1, -1, 2, -2, 3, -3, ...$

6#) Find the limits of the following sequences or state they do not exist. (You need not justify your answers.)

(i) $\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, ...,$

(ii) $\frac{3}{2}, -\frac{4}{3}, \frac{5}{4}, -\frac{6}{5}, ...,$

(iii) $-1, -2, -3, -4, ...,$

(iv) $-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, ...,$

(v) $2, 2.2, 2.22, 2.222, ...$

7#) Using, if necessary, a calculator evaluate at least the first 6 terms of each of the following sequences. Can you guess what the limit is in each case? If a limit is not readily apparent calculate, if possible, a few terms of the sequence with n various small powers of 10.

(i) $a_1 = 1, \qquad 2a_{n+1} = a_n + 3 \quad$ for each $n \geq 1$;

(ii) $b_1 = 1, \qquad b_{n+1} = 1 + \frac{1}{1+b_n} \quad$ for each $n \geq 1$;

(iii) $c_n = n - \sqrt{n(n-1)} \quad$ for each $n \geq 1$;

(iv) $d_n = n^{1/n} \quad$ for each $n \geq 1$;

(v)** $e_1 = \sqrt{2}, \qquad e_{n+1} = \sqrt{2 + e_n} \quad$ for each $n \geq 1$;

(vi)** $f_n = 2^{n+1}\sqrt{2 - e_n} \quad$ for each $n \geq 1$, where e_n is as in part (iii).

8#) From calculating the value of $\sqrt{n+1} - \sqrt{n-1}$ for a few values of n you might guess that the sequence $\left\{\sqrt{n+1} - \sqrt{n-1}\right\}_{n\in\mathbb{N}}$ converges with limit 0. To prove this you would have to verify the definition of convergence which we will do in a later question.

For now, take each value of ε below and calculate the smallest value $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\left|\sqrt{n+1} - \sqrt{n-1}\right| < \varepsilon \qquad \text{for all } n \geq N.$$

(i) $\varepsilon = \frac{1}{10}, \qquad$ (ii) $\varepsilon = \frac{1}{20}, \qquad$ (iii) $\varepsilon = \frac{1}{40}, \qquad$ (iv) $\varepsilon = \frac{1}{80}$.

As we half the value of ε what happens to N?

(You may assume that $\left\{\sqrt{n+1} - \sqrt{n-1}\right\}_{n\in\mathbb{N}}$ is a decreasing sequence, though you might also try to prove this.)

9) In each example below verify the definition of limit. So, assume that an $\varepsilon > 0$ is given. Explain how the Archimedean Principle can be used to find $N = N(\varepsilon) \in \mathbb{N}$ such that if $n \geq N$ then $|x_n - \ell| < \varepsilon$.

Also, in each case below, if ε is halved how does $N(\varepsilon)$ change?

(i) $\left\{\frac{n-1}{2n}\right\}_{n\in\mathbb{N}}$ has limit $\frac{1}{2}$, (ii) $\left\{\frac{(-1)^n}{n}\right\}_{n\in\mathbb{N}}$ has limit 0,

**(iii) $\left\{\frac{2n+1}{3n-1}\right\}_{n\in\mathbb{N}}$ has limit $\frac{2}{3}$, (iv) $\left\{\sqrt{\frac{1}{n}}\right\}_{n\in\mathbb{N}}$ has limit 0.

10) By verifying the definition of convergence prove that the following series converge and find their limits..

(i) $\left\{\frac{n^2-n}{n^2+n}\right\}_{n\in\mathbb{N}}$ (ii) $\left\{\frac{5^n-1}{5^n+1}\right\}_{n\in\mathbb{N}}$ (iii) $\left\{\frac{5^n-3^n}{5^n+3^n}\right\}_{n\in\mathbb{N}}$.