153 Problem Sheet 1

All questions should be attempted. Those marked with a ** must be handed in for marking by your supervisor. Hopefully the supervisor will have time to cover at least the questions marked with a * or **. Questions marked with a # will be discussed in the problems class. Those marked with H are slightly harder than the others.

1#) Prove or disprove the following statements:

(i) For all $a, b \in \mathbb{R}$ if a and b are rational then $a + b$ is rational;
(ii) For all $a, b \in \mathbb{R}$ if a and b are irrational then $a + b$ is irrational.

2) (i) You know from course 112 that $\sqrt{2}$ is irrational. Use this fact to prove that $\sqrt{3 + 2\sqrt{2}}$ is irrational.

Hint: Use proof by contradiction, so assume $\sqrt{3 + 2\sqrt{2}}$ is rational.

(ii) Let $x = \sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}$. Calculate x^2. Is x irrational?

3**) (i) Prove that $\sqrt{6}$ is irrational.

(Hint: Use the same method as you used in course 112 to prove that $\sqrt{2}$ is irrational.)

(ii) Use part (i) and proof by contradiction to prove that $\sqrt{2}+\sqrt{3}$ is irrational.

4#) Find a polynomial with integer coefficients for which $\sqrt{2 + \sqrt{3 + \sqrt{5}}}$ is a root.

5H#) (i) Prove that no rational number satisfies $x^4 - 16x^2 + 4 = 0$.

(Hint: Use proof by contradiction **exactly** as you did in the proof that $\sqrt{2}$ is irrational.)

(ii) Use part (i) to prove that $\sqrt{5} + \sqrt{3}$ is irrational.

6*) (i) Show that the real number with decimal expansion $113.254\overline{467}$ (that is, with 467 repeated) is rational. (Hint: Consider the sum of a certain geometric progression.)

(ii) Calculate, *by hand*, the decimal expansion of the rational number $\frac{2041}{495}$. By looking at the remainders at each step show that the decimal expansion must, by necessity, repeat.

(iii) Verify informally (so no *rigorous* proof is required) that a real number is rational if, and only if, its decimal expansion either terminates or repeats.

(iv) Verify informally that if a and b are real numbers, with $a < b$ then there exists a rational number c with $a < c < b$ and there exists an irrational number d with $a < d < b$.

(v) Verify informally that if a and b are real numbers, with $a < b$ then there exists *infinitely many* rational numbers c with $a < c < b$ and there exists *infinitely many* irrational numbers d with $a < d < b$.

7#) Using the Properties 1-5 of the real numbers, and the results derived from them in the notes, give justifications for each step in the argument that follows:

Claim $(-1)(-1) = 1$

Solution.

Put your reasons below

$$0 = 1 + (-1),$$

$$-1 \times 0 = -1(1 + (-1)),$$

$$0 = (-1)1 + (-1)(-1),$$

$$0 = -1 + (-1)(-1),$$

$$1 + 0 = 1 + (-1 + (-1)(-1)),$$

$$1 = (1 + (-1)) + (-1)(-1),$$

$$1 = 0 + (-1)(-1),$$

$$1 = (-1)(-1).$$

8) Prove the result stated in the notes as Derived Property D:

$$\text{For all } a, b \in \mathbb{R}, \ \ (-a)\, b = -\,(ab)\,.$$

9) Use Properties 1-9 of the real numbers to prove that

$$\text{if } c \leq 0 \text{ and } a \geq b \text{ then } ac \leq bc.$$

10#) Find the *glb*s and the *lub*s of the following subsets of $\mathbb{R}$, or state that the *glb*s and the *lub*s do not exist. (You need not justify your answers.)

(i) $\mathbb{N}_0$, (ii) $\mathbb{Q}$, (iii) $\{6\}$, (iv) $\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ...\right\}$, (v) $(-1, 3]$, (vi) $\{2, 2.2, 2.22, 2.222, 2.2222, ...\}$.

11*) Draw graphs of the following functions

i) $y = |x-3|$ ii) $y = |x+1|$
iii) $y = |x-3| + |x+1|$ iv) $y = |x+1| - |x-3|$
v) $y = 2\,|x+1| - 3\,|x-1|$.

12#) Find the *glb*s and the *lub*s of the following subsets of $\mathbb{R}$, or state that the *glb*s and the *lub*s do not exist. (You need not justify your answers.)

(i) $\{x : 1 < |x-3| < 4\}$,

(ii) $\{|x-3| : 1 < x < 4\}$,

(iii) $\{x : 0 \leq |x+1| - |x-3| \leq 4\}$,

(iv) $\{|x+1| - |x-3| : 0 \leq x \leq 4\}$.

Make sure you understand how the sets in (i) and (ii) differ from each other, and similarly for those in (iii) and (iv).

13#) Use the definition of *lub* to prove that 4 is the *lub* of $\{1, 2, 3, 4\}$.