Appendix 1.2

1. Do **not** mix up the definitions for $\lim_{x\to a} f(x) = c$ with a finite limit $c \in \mathbb{R}$ and $\lim_{x\to a} f(x) = +\infty$ or $-\infty$.

   For $\lim_{x\to a} f(x) = +\infty$ we do **not** look at $|f(x) - \infty|$ for x close to a since ∞ is not a real number and thus $f(x) - \infty$ has no meaning.

   Do **not** use the quotient rule when trying to prove that a limit is infinite.

   Example 1.2.13 *We* ***cannot*** *say*

   $$\lim_{x\to 1} \frac{1}{(x-1)^2} = \frac{1}{\lim_{x\to 1} (x-1)^2}$$

   because $\lim_{x\to 1} (x-1)^2 = 0$ *which is excluded from the Quotient Rule. If you continue you get*

   $$\lim_{x\to 1} \frac{1}{(x-1)^2} = \frac{1}{\lim_{x\to 1} (x-1)^2} = \frac{1}{0}$$

   which is ***not*** *defined (and in particular it is* ***not*** *equal to* $+\infty$*).*

2. **Proof of the Sum Rule for limits.** If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then

   $$\lim_{x\to a} (f+g)(x) = L + M.$$

   Let $\varepsilon > 0$ be given.

   From the definition of $\lim_{x\to a} f(x) = L$ we find $\delta_1 > 0$ such that

   $$0 < |x-a| < \delta_1 \implies |f(x) - L| < \varepsilon/2. \qquad (11)$$

   From the definition of $\lim_{x\to a} g(x) = M$ we find $\delta_2 > 0$ such that

   $$0 < |x-a| < \delta_2 \implies |g(x) - M| < \varepsilon/2. \qquad (12)$$

   Let $\delta = \min(\delta_1, \delta_2)$. Assume $0 < |x-a| < \delta$. For such x both (11) and (12) hold. Thus

   $$\begin{aligned} |(f(x)+g(x)) - (L+M)| &= |f(x) - L + g(x) - M| \\ &\le |f(x) - L| + |g(x) - M| \\ &\qquad \text{by triangle inequality,} \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Thus we have verified the definition of

$$\lim_{x\to a} (f+g)(x) = L + M.$$

■

3. **Limits of Rational Functions.** In the lectures we have shown that the **limit** of a rational function at a point is the **value** of the rational function at that point, *provided that value is defined.* So assume we are given a rational function $r(x) = p(x)/q(x)$ and a point a for which $q(a) \neq 0$ (so $r(a)$ is defined). If we are required to verify the definition that $\lim_{x\to a} r(x) = r(a)$ we need examine

$$|r(x) - r(a)| = \left|\frac{p(x)}{q(x)} - \frac{p(a)}{q(a)}\right| = \left|\frac{p(x)\,q(a) - p(a)\,q(x)}{q(x)\,q(a)}\right|.$$

The numerator here, $p(x)\,q(a) - p(a)\,q(x)$, is a polynomial that is zero when $x = a$. This means is has a factor of $x - a$, i.e.

$$p(x)\,q(a) - p(a)\,q(x) = (x-a)\,m(x)\,,$$

for some polynomial $m(x)$. Then

$$|r(x) - r(a)| = |x - a| \left|\frac{m(x)}{q(x)\,qv\,(a)}\right|.$$

The verification proceeds by assuming $0 < |x-a| < \delta$, which is applied to the first term. Then assume $\delta \leq C$ for some constant C for which the second term $m(x)/q(x)\,q(a)$ is defined on $|x-a| < C$. Then an upper bound $M > 0$ is found for this term, i.e.

$$\left|\frac{m(x)}{q(x)\,q(a)}\right| \leq M$$

for $|x-a| < C$. It suffices then to choose $\delta = \min(C, \varepsilon/M)$.