MATH20101 Real Analysis

These notes contain the statements of all results and examples but not necessarily the proofs. For these, you will have to attend the lectures.

Part 1.1. Limits of functions

v1. 2019-20

Introduction

Example 1.1.1 *What happens to* $f : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$, *given by*

$$f(x) = \frac{x^2 - 1}{x^2 - x}, x \neq 1, 0,$$

as x *gets close to 1?*

If we substitute $x = 1$ we get

$$f(1) = \frac{0}{0},$$

which is **undefined**. Instead we might try to substitute numbers close to 1:

x	$f(x)$
1.1	1.90909
1.01	1.99009
1.001	1.99900
1.0001	1.99990

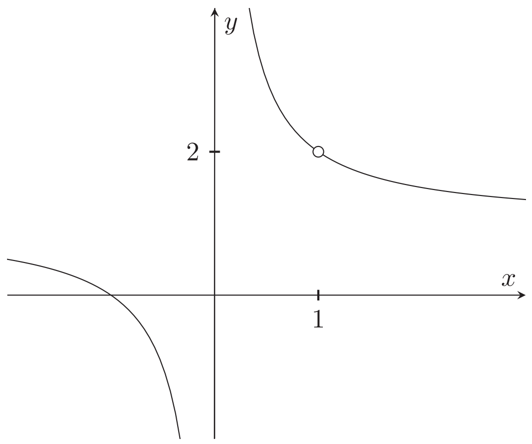
x	$f(x)$
0.9	2.11111
0.99	2.01010
0.999	2.00100
0.9999	2.00010

It appears that if x is "very close" to 1 then $f(x)$ is "very close" to 2. In fact, factorization shows

$$\frac{x^2 - 1}{x^2 - x} = \frac{(x-1)(x+1)}{x(x-1)} = 1 + \frac{1}{x},$$

though, as an equation, this still **only** holds for $x \neq 0, 1$, since we are **never allowed to divide by zero**.

Graphically we have:



where the point $(1, 2)$ is omitted from the graph. It is obvious from the graph that as x approaches 1 then $f(x)$ approaches 2.

Limit of a function.

Intervals and neighbourhoods.

Definition 1.1.2 *Define the **open** intervals $(a, b) = \{x : a < x < b\}$ and the **closed** intervals $[a, b] = \{x : a \leq x \leq b\}$ along with the obvious **open-closed** and **closed-open** intervals, where $a, b \in \mathbb{R}$.*

I leave it to the student to define the ***open-closed*** intervals $(a, b]$ and ***closed-open*** intervals $[a, b)$.

Important Aside: It is important that you learn and memorise *all* the definitions in this course. You should attempt to memorise them so well that you can write them down with no thought.

> **Advice for exams:** I consider that when, in an exam paper, I ask you to give a definition then that is the opportunity for you to gain easy marks.

Why should you learn all the definitions? How can you verify that a subset of the real numbers is an open interval if you don't know what an open interval is? And if you are told that a subset of the real numbers is an open interval how can you use that information if you don't know what an open interval is?

Definition 1.1.3 • *A* ***neighbourhood*** *of a is an open interval $(a-\delta, a+\delta)$ for some $\delta > 0$. (Often in lectures I will write* ***nbhd*** *instead of the longer word, neighbourhood).*

• *A* ***deleted neighbourhood*** *of a is a neighbourhood of a with a deleted, i.e. removed, so $(a-\delta, a+\delta) \setminus \{a\}$ for some $\delta > 0$. It can be looked upon as the union of two open intervals, i.e. $(a-\delta, a) \cup (a, a+\delta)$ for some $\delta > 0$.*

Recall that the *modulus* of a real number x is given by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Do remember that this means that $|x| \geq 0$ for all $x \in \mathbb{R}$ and that $|x|$ measures the *magnitude* of x, i.e. the *distance* of x from the origin without sign. In our case we can write a **neighbourhood** as

$$(a-\delta, a+\delta) = \{x : |x-a| < \delta\},$$

and a **deleted neighbourhood** as

$$(a-\delta, a) \cup (a, a+\delta) = \{x : 0 < |x-a| < \delta\}.$$

Assumptions

Assumption Given a function $f : A \to \mathbb{R}$ and a point $a \in \mathbb{R}$, which may or may not lie in A, we will assume that there exists a deleted neighbourhood of a lying inside A, i.e.

$$\exists\, \delta > 0 : (a-\delta, a) \cup (a, a+\delta) \subseteq A.$$

Example 1.1.4 *If*

$$f(x) = \frac{x^2-1}{x^2-x}, \;\; A = \mathbb{R} \setminus \{0, 1\},$$

and $a = 1$ then the assumption is satisfied (even though $a \notin A$).

Proof We need only find a $\delta > 0$ and we can take $\delta = 1$. This works since

$$(1-1, 1) \cup (1, 1+1) = (0, 1) \cup (1, 2) \subseteq \mathbb{R} \setminus \{0, 1\}.$$

■

Example 1.1.5 *If*

$$f(x) = x,\ A = [0,1]\,,$$

and $a = 1$ *then the assumption is* ***not*** *satisfied (even though* $a \in A$*).*

Proof For *any* $\delta > 0$ we note that $(1-\delta, 1) \cup (1, 1+\delta)$ contains a number greater than 1 (i.e. $1 + \delta/2$) in which case

$$(1-\delta, 1) \cup (1, 1+\delta) \nsubseteq [0,1]\,.$$

■

Definition of limit at a finite point.

Definition 1.1.6 *Let* $f : A \to \mathbb{R}$ *be a function whose domain contains a deleted neighbourhood of* $a \in \mathbb{R}$*. The* ***limit of the function*** f ***at*** a ***is*** L *if, and only if, for all* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that if* $0 < |x - a| < \delta$ *then* $|f(x) - L| < \varepsilon$*. That is:*

$$\forall \varepsilon > 0, \exists\, \delta > 0, \forall x \in A, 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon. \tag{1}$$

(Recall that in logic we read "$p \Rightarrow q$*" as "if* p *holds then* q *will hold".)*

We write

$$\lim_{x \to a} f(x) = L.$$

But we may also write "$f(x) \to L$ *as* $x \to a$*", which we read as "*$f(x)$ *tends to* L *as* x *tends to* a*". (Often in lectures I will write* $\mathcal{L}_{x\to a} f(x) = L$*).*

Important Observation The *order* of the quantifiers in (1) is of great importance. The definition says there exists $\delta > 0$ such that for all x something happens. This means that δ has to **not** depend on x.

It is also important that you note that in the definition we have

$$\mathbf{0 < |x - a|\,,\ \text{i.e. } x \neq a}.$$

That is, the limit of a function is calculated *at* a point a, by looking at the value of the function at points x ***near, but not equal, to*** the point a.

Examples of limits at finite points.

Example 1.1.7 *Use the ε - δ definition to prove that*

$$\lim_{x\to 1}\left(x^2-1\right)=0.$$

Solution Let $\varepsilon>0$ be given. Choose $\delta=\min(1,\varepsilon/3)$. (This means that δ satisfies **two** inequalities, $\delta\le 1$ **and** $\delta\le\varepsilon/3$.) Assume that x satisfies $0<|x-1|<\delta$.

Then, for such x,

$$\begin{aligned}|f(x)-L| &= \left|\left(x^2-1\right)-0\right|=\left|x^2-1\right|\\ &= |x-1|\,|x+1|<\delta\,|x+1|\,.\end{aligned}$$

Yet $|x-1|<\delta\le 1$. So

$$\begin{aligned}|x+1| &= |(x-1)+2|\le|x-1|+2 \qquad \text{by triangle inequality}\\ &\le 1+2=3.\end{aligned}$$

Combining,

$$|f(x)-L|<\delta\,|x+1|<3\delta\le\varepsilon$$

since $\delta\le\varepsilon/3$. Hence we have shown that given any $\varepsilon>0$ we can find (and the best way to show we can find something is to give an explicit example of it) a $\delta>0$ such that assuming $0<|x-1|<\delta$ we can do some mathematics to deduce $|f(x)-L|<\varepsilon$. That is, we have verified the definition that $\lim_{x\to 1}\left(x^2-1\right)=0$. ■

The point of this example is, how did we find δ?

Example 1.1.8 *Use the ε - δ definition to prove that*

$$\lim_{x\to 2}\left(x^3+x^2-4x\right)=4.$$

Rough work Assume $0<|x-2|<\delta$ *with* $\delta>0$ *to be found.* Consider

$$f(x)-L=\left(x^3+x^2-4x\right)-4=x^3+x^2-4x-4.$$

We need to factor this, but we note that $x=2$ is a root (why?) so

$$x^3+x^2-4x-4=(x-2)\left(ax^2+bx+c\right),$$

for some coefficients a, b and c. Do **NOT** use long division of polynomials to find the coefficients, it takes too long. Simply equate powers on both sides to see that

$$x^3 + x^2 - 4x - 4 = (x-2)\left(x^2 + 3x + 2\right).$$

Demanding $\delta \leq 1$ gives $|x-2| < \delta \leq 1$. You can open this up as

$$-1 < x - 2 < 1, \quad \text{i.e.} \quad 1 < x < 3,$$

so $|x| < 3$. Alternatively, the triangle inequality gives

$$|x| = |(x-2) + 2| \leq |x-2| + 2 \leq 1 + 2 = 3.$$

Again using the triangle inequality,

$$\left|x^2 + 3x + 2\right| \leq |x|^2 + 3|x| + 2 < 20.$$

Thus

$$|f(x) - L| = \left|(x-2)\left(x^2 + 3x + 2\right)\right| < 20\delta.$$

We require this to be $\leq \varepsilon$ for the definition of limit to be satisfied, so we demand $\delta \leq \varepsilon/20$.

Combine the two requirements $\delta \leq 1$ and $\delta \leq \varepsilon/20$ as one, namely $\delta = \min(1, \varepsilon/20)$.

End of Rough Work.

Note We chose 1 as an upper bound on δ only because it is the 'simplest' positive number. In an example below we see that we need choose an upper bound strictly less than one (and the 'simplest' positive number strictly less than 1 is $1/2$.)

Solution Let $\varepsilon > 0$ be given. Choose $\delta = \min(1, \varepsilon/20)$. Assume x satisfies $0 < |x-2| < \delta$.

Then, for such x,

$$|f(x) - L| = \left|(x-2)\left(x^2 + 3x + 2\right)\right| < \delta\left|x^2 + 3x + 2\right|.$$

Yet $|x-2| < \delta \leq 1$ implies, by the triangle inequality,

$$|x| = |(x-2) + 2| \leq |x-2| + 2 \leq 1 + 2 = 3.$$

Again using the triangle inequality,

$$\left|x^2 + 3x + 2\right| \leq |x|^2 + 3|x| + 2 \leq 20.$$

Combining,

$$|f(x) - L| < \delta \left|x^2 + 3x + 2\right| \le 20\delta \le 20\left(\frac{\varepsilon}{20}\right) = \varepsilon.$$

Hence we have verified the definition of

$$\lim_{x\to 2}\left(x^3 + x^2 - 4x\right) = 4.$$

$\blacksquare$

Note on the LOGIC of a proof. In a proof we start with the assumptions and end with the required conclusion. Compare this with the rough work where we started with the conclusion, $|f(x) - L| < \varepsilon$, and worked out what conditions δ must satisfy to ensure this conclusion.

A proof does **not** start with what you want to prove followed by mathematical reasoning leading to a true statement. For recall from MATH10101 the truth table for $p \Rightarrow q$. If p is FALSE and q TRUE then the truth table gave $p \Rightarrow q$ as TRUE. Thus it is possible to start with a false assumption, p, apply mathematical reasoning (represented by $\Rightarrow$) and deduce a true statement, q. Thus ending a mathematical argument with a true statement does **not** imply that you started with a true statement.

Example 1.1.9 *By verifying the ε - δ definition show that*

$$\lim_{x\to -1}\left(5x^3 + 16x^2 + 13x + 3\right) = 1.$$

Solution left to Tutorial
Rough work Assume $0 < |x - (-1)| < \delta$ with $\delta > 0$ to be found. Consider

$$f(x) - L = \left(5x^3 + 16x^2 + 13x + 3\right) - 1 = 5x^3 + 16x^2 + 13x + 2.$$

We need to factor this, but we note that $x = -1$ is a root (why?). From this we get

$$5x^3 + 16x^2 + 13x + 2 = (x + 1)\left(5x^2 + 11x + 2\right).$$

Demand $\delta \le 1$ in which case $|x + 1| < \delta \le 1$. This opens out as $-1 < x + 1 < 1$, i.e. $-2 < x \le 0$. We require **an** upper bound on $|5x^2 + 11x + 2|$ for such x. Do not waste time by finding the best upper bound for *any* will suffice. I suggest using the triangle inequality.

The restriction $-2 < x \le 0$ implies $|x| < 2$. The **triangle inequality** then gives

$$\left|5x^2 + 11x + 2\right| \le 5|x|^2 + 11|x| + 2 < 20 + 22 + 2 = 44.$$

Thus

$$|f(x) - L| = |x+1| \left|5x^2 + 11x + 2\right| < 44\delta$$

which we demand is $\leq \varepsilon$.

Combine the two demands $\delta \leq 1$ and $\delta \leq \varepsilon/44$ as one, namely

$$\delta = \min\left(1, \frac{\varepsilon}{44}\right).$$

End of Rough Work.

Solution Let $\varepsilon > 0$ be given. Choose $\delta = \min(1, \varepsilon/44)$. Assume x satisfies $0 < |x+1| < \delta$. Then, for such x, as seen in the Rough Work,

$$f(x) - L = (x+1)\left(5x^2 + 11x + 2\right).$$

Because $\delta \leq 1$ we have $|x+1| < \delta \leq 1$ and thus $-2 < x \leq 0$ and $|x| < 2$. Then, by the triangle inequality,

$$\left|5x^2 + 11x + 2\right| \leq 5|x|^2 + 11|x| + 2 < 20 + 22 + 2 = 44.$$

Thus

$$\begin{aligned} |f(x) - L| &= |x+1| \left|5x^2 + 11x + 2\right| < 44\delta \\ &\leq 44\left(\frac{\varepsilon}{44}\right) = \varepsilon. \end{aligned}$$

Hence we have verified the definition that $\lim_{x \to -1}\left(5x^3 + 16x^2 + 13x + 3\right) = 1$. ■

Note Be very careful. We can expand $|x+1| < 1$ as $-2 < x \leq 0$. In looking for an upper bound on $|5x^2 + 11x + 2|$ it would be *wrong* to evaluate $5x^2 + 11x + 2$ only at the end points. To say that $-2 < x \leq 0$ implies

$$5x^2 + 11x + 2\Big|_{x=-2} \leq 5x^2 + 11x + 2 \leq 5x^2 + 11x + 2\Big|_{x=0},$$

would lead to

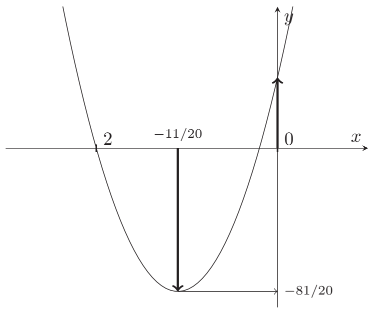
$$5(-2)^2 + 11 \times -2 + 2 \leq 5x^2 + 11x + 2 \leq 5 \times 0^2 + 11 \times 0 + 2,$$

which simplifies to

$$0 \leq 5x^2 + 11x + 2 \leq 2.$$

This is saying that $5x^2 + 11x + 2$ is never negative in the interval $-2 \leq x \leq 0$.

Yet, at $x=-1$, the quadratic is -4 and the graph shows it to be even more negative!.



For a more detailed analysis (which is **not** required) complete the square so

$$5x^2+11x+2=5\left(x+\frac{11}{10}\right)^2-\frac{81}{20}$$

when

$$5x^2+11x+2\big|_{x=-11/10}=-\frac{81}{20}.$$

It is then not so hard to show that

$$\max_{-2\le x\le 0}\left|5x^2+11x+2\right|=\frac{81}{20}.$$

This shows how weak is our bound $|5x^2+11x+2|\le 44$, but this is irrelevant, **any** bound will suffice as long as you can justify it.

Return now to the example which introduced the idea of a limit.

Example 1.1.1 *By verifying the ε - δ definition show that*

$$\lim_{x\to 1}\frac{x^2-1}{x^2-x}=2.$$

Rough Work Assume $0<|x-1|<\delta$ with $\delta>0$ to be found. Then

$$\begin{aligned}|f(x)-L| &= \left|\frac{x^2-1}{x^2-x}-2\right|=\left|\frac{x^2-1-2\left(x^2-x\right)}{x^2-x}\right| \\ &= \left|\frac{-x^2+2x-1}{x^2-x}\right|=\left|-\frac{(x-1)^2}{x\left(x-1\right)}\right| \\ &= \frac{|x-1|}{|x|}<\frac{\delta}{|x|}.\end{aligned}$$

We need to ensure that x does not get too close to 0 (for then $1/|x|$ would get very large). Demand $\delta \leq 1/2$ for then $0 < |x-1| < \delta \leq 1/2$ which opens out as

$$-\frac{1}{2} < x - 1 < \frac{1}{2} \quad \text{and this implies} \quad x > \frac{1}{2}.$$

Thus

$$\left| \frac{x^2-1}{x^2-x} - 2 \right| < \frac{\delta}{|x|} < 2\delta.$$

We now demand that this is $< \varepsilon$, i.e. $2\delta \leq \varepsilon$, for then

$$\left| \frac{x^2-1}{x^2-x} - 2 \right| < \frac{\delta}{|x|} < 2\delta \leq \varepsilon, \quad \text{i.e.} \quad \left| \frac{x^2-1}{x^2-x} - 2 \right| < \varepsilon$$

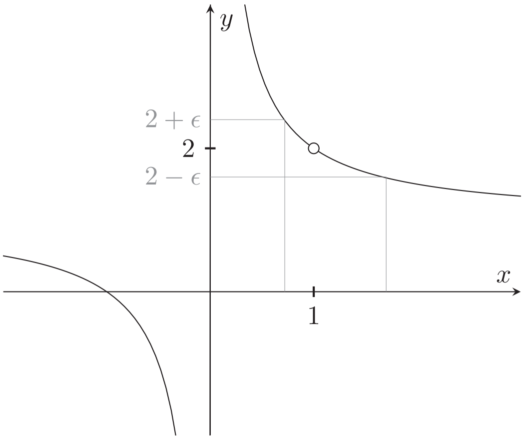
as required for the ε - δ definition to hold. Combine the two requirements $\delta \leq 1/2$ and $\delta \leq \varepsilon/2$ as one, namely $\delta = \min(1/2, \varepsilon/2)$.

End of Rough Work.

Solution left to students. ■

Note 1 If there are no reasons why I cannot, I will always assume $\delta \leq 1$. In this last example we could not use the bound $\delta \leq 1$ for if we did, the range on x would be $0 < x < 2$, which would mean that x can be as close to 0 as we like. And if x is close to 0 then $1/|x|$ is large. In fact, $1/|x|$ is unbounded for $0 < x < 2$. Instead we demand $\delta \leq 1/2$, choosing $1/2$ as the 'simplest' positive number strictly less than 1. Then $\delta \leq 1/2$ implies $1/2 \leq x \leq 3/2$ and for such x the reciprocal $1/|x|$ is bounded.

Note 2 The graph of $y = (x^2 - 1) / (x^2 - x)$:



We see from this picture that the actual interval of $x : |f(x) - 2| \leq \varepsilon$ is not a symmetric deleted neighbourhood of 1. So the set of $x : 0 < |x - 1| < \delta$ is a *proper* subset of all $x : |f(x) - 2| \leq \varepsilon$.

Example 1.1.10 *By verifying the* ε - δ *definition show that*

$$\lim_{x \to 2} \frac{x^2 + 2x + 2}{x + 3} = 2.$$

Solution On Problem Sheet. This requires bounding

$$\left| \frac{x + 2}{x + 3} \right|$$

from above for $|x - 2| \leq 1$. ■

Advice for the exams.

1 It is important that you know how to *factorise polynomials*. If one of the factors is linear, i.e. of the form $x - a$, there is **no** need to use long division of polynomials. Instead use the method of equating coefficients.

2 It is important that you can *manipulate inequalities containing modulus signs*.

3 It is important that you can give *upper bounds* for *polynomials*, such as $x^2 + 3x + 2$, or *rational functions*, such as $(x + 2)/(x + 3)$ above, with x restricted to an interval of the form $|x + c| < \delta$.

Note 1 Given a function f and a point a to prove that $\lim_{x\to a} f(x) = L$ you have to verify the ε - δ definition. Look upon this as a game where, given $\varepsilon > 0$, you have to find the δ.

Note 2 In almost all cases the δ you find will be the minimum of a constant and a function of ε.

Note 3 As has been stressed above, it is not important that $\delta \leq 1$ in the verification of $\lim_{x\to 1} (x^2 - 1) = 0$. The number 1 is chosen only because it is the 'simplest' positive number. You might instead have demanded that $\delta \leq 2$. This would mean $|x - 1| < 2$ which would imply $|x + 1| \leq 4$ (check that this is so). Then we would choose $\delta = \min(2, \varepsilon/4)$. There are infinitely many possible choices for δ, but to verify the ε - δ definition we only have to find one.

Note 4 We can extract a common theme of the examples above. When verifying the ε - δ definition of $\lim_{x\to a} f(x) = L$ we have seen that when $0 < |x - a| < \delta$,

$$|f(x) - L| = |x - a| \, |G(x)| < \delta \, |G(x)| ,$$

for some function $G(x)$. We need to bound this function. To do this we restrict $\delta \leq C$ for some constant C, normally 1, but chosen such that $G(x)$ is defined and bounded for all x satisfying $0 < |x - a| < C$. Let M be any upper bound for $G(x)$, so

$$\sup_{0<|x-a|<C} |G(x)| \leq M.$$

Then for such x we have

$$|f(x) - L| < \delta \, |G(x)| \leq \delta \sup_{0<|x-a|<C} |G(x)| \leq \delta M$$

which we demand to be $\leq \varepsilon$. Combining the demands on δ gives $\delta = \min(C, \varepsilon/M)$.

Note 5 The δ **never** depends on x. In an example above we have

$$\left| \frac{x^2 - 1}{x^2 - x} - 2 \right| < \frac{\delta}{|x|}.$$

It would be **wrong** to demand that $\delta/|x| < \varepsilon$, i.e. $\delta < \varepsilon \, |x|$, for this bound depends on x. Instead we spent the time to show that $1/|x| < 2$ for then $\delta/|x| < 2\delta$ and we can then demand that $2\delta \leq \varepsilon$.

Alternative definition of a limit of a function.

It is possible to relate the definition of limit given here to the idea of limits of sequences introduced in MATH10242 Sequences and Series.

Definition 1.1.11 *Let $f : A \to \mathbb{R}$ be a function whose domain contains a deleted neighbourhood of $a \in \mathbb{R}$. Then*

$$\lim_{x\to a} f(x) = L$$

iff for all sequences $\{x_n\}_{n\geq 1} \subseteq A$ for which $x_n \neq a$ for all $n \geq 1$ and $\lim_{n\to\infty} x_n = a$ we have

$$\lim_{n\to\infty} f(x_n) = L.$$

Note In this definition we replace $x \to a$ by the set of all sequences $x_n \to a$. But the point of looking at the limit of f at a rather than $f(a)$ is that $f(a)$ might not be defined. Yet when looking at sequences we need that $f(x_n)$ be defined for all $n \geq 1$ and so we demand $x_n \neq a$ for all $n \geq 1$.

Proof of the equivalence of the definitions is not given in this course, see Appendix. ■

That we do not prove the equivalence does not mean you should not learn this second definition; it is useful. For example, this second definition can be used to proving that a limit does **not** exist.

The following version of the definition does not give a label to the limit.

Definition 1.1.12 *Let $f : A \to \mathbb{R}$ be a function whose domain contains a deleted neighbourhood of $a \in \mathbb{R}$. Then $\lim_{x\to a} f(x)$ exists iff there exists L : for all sequences $\{x_n\}_{n\geq 1} \subseteq A\backslash\{a\}$, $\lim_{n\to\infty} x_n = a \implies \lim_{n\to\infty} f(x_n) = L$.*

Consider one of these implications,

$$\lim_{x\to a} f(x) \text{ exists } \implies \qquad (2)$$
$$\exists\, L, \forall \{x_n\}_{n\geq 1} \subseteq A \setminus \{a\} : \lim_{n\to\infty} x_n = a \implies \lim_{n\to\infty} f(x_n) = L.$$

Recall from Mathematical Logic that the *contrapositive* of an implication '$P \Rightarrow Q$' is the logically equivalent (i.e. they have the same truth tables) 'not$Q \Rightarrow$ not$P$'.

The negation of the right hand side of (2) is

$$\forall L, \exists \{x_n\}_{n\geq 1} \subseteq A \setminus \{a\} : \lim_{n\to\infty} x_n = a \text{ and } \lim_{n\to\infty} f(x_n) \neq L. \qquad (3)$$

It could be that $\lim_{n\to\infty} f(x_n) \neq L$ because $\lim_{n\to\infty} f(x_n)$ does not exist, i.e. $\{f(x_n)\}_{n\geq 1}$ is a divergence sequence.

If $\{f(x_n)\}_{n\geq 1}$ is a convergent sequence for all $\{x_n\}_{n\geq 1} \subseteq A \setminus \{a\}$ with $\lim_{n\to\infty} x_n = a$ then (3) holds only if the sequences $\{f(x_n)\}_{n\geq 1}$ do **not** all have the same limit for all sequences.

Hence the contrapositive of (2) is

Corollary 1.1.13 *Let $f : A \to \mathbb{R}$ be a function whose domain contains a deleted neighbourhood of $a \in \mathbb{R}$. If either there exists a sequence $\{x_n\}_{n\geq 1} \subseteq A$ for which*

- *$x_n \neq a$ for all $n \geq 1$ and $\lim_{n\to\infty} x_n = a$,*
- *$\lim_{n\to\infty} f(x_n)$ does not exist*

or there exist two sequences $\{x_n\}_{n\geq 1} \subseteq A$ and $\{y_n\}_{n\geq 1} \subseteq A$ for which

- *$x_n \neq a$ for all $n \geq 1$ and $\lim_{n\to\infty} x_n = a$,*
- *$y_n \neq a$ for all $n \geq 1$ and $\lim_{n\to\infty} y_n = a$,*
- *$\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$*

then $\lim_{x\to a} f(x)$ does not exist.

To understand the theory of limits it is important to have examples of functions *without* a limit at a point. Such an example is

Example 1.1.14

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$$

does not exist.

Solution Left to the student to apply Corollary 1.1.13, see a Problem Sheet. In the Appendix there is an alternative proof using proof by contradiction.

The graph for $\sin(\pi/x)$ is