MATH10242 Sequences and Series: Solutions 3, to exercises for week 4 Tutorials,

Question 1: Which of the following sequences converge (and to what number)? Justify your answers.

(a) $(1 - \dfrac{3n^3 + n^2}{2n^3})_{n\geq 1}$

(b) $(1 - \dfrac{3n^2 + n^3}{2n^2})_{n\geq 1}$

(c) $(\sqrt{n^2+1} - n)_{n\geq 1}$ [*Hint: Use the ideas we used for* $\sqrt{n+2} - \sqrt{n}$]

(d) $(\sqrt{2n} - \sqrt{n})_{n\geq 1}$

(e) $(3^{-n})_{n\in\mathbb{N}}$

(f) $\left(\dfrac{n^3}{3^n + 4^n}\right)_{n\geq 1}$

Solution: (a) It is a good idea to manipulate the messy function of n into something nicer. The key idea is to divide top and bottom by n^3,

$$\frac{3n^3 + n^2}{2n^3} = \frac{3 + n^{-1}}{2} = \frac{3}{2} + \frac{1}{2n}.$$

Thus

$$1 - \frac{3n^3 + n^2}{2n^3} = -\frac{1}{2} - \frac{1}{2n}.$$

Now it is easy—if we pick $N = [1/2\varepsilon]+1$ then (with the usual manipulation)) $N > 1/2\varepsilon$ and $\frac{1}{2N} < \varepsilon$. For $n \geq N$ the earlier computations show that

$$\left|\left(1 - \frac{3n^3 + n^2}{2n^3}\right) + \frac{1}{2}\right| = \left|\left(-\frac{1}{2} - \frac{1}{2n}\right) + \frac{1}{2}\right| = \left|-\frac{1}{2n}\right| = \frac{1}{2n} \leq \frac{1}{2N} < \varepsilon.$$

In other words $\lim\limits_{n\to\infty} a_n = -1/2$.

Remark: By the time you read this we will have done the Algebra of Limits Theorem. So you could use that and argue as follows:

We know from the lectures that $\lim_{n\to\infty} 1/n = 0$. Hence

$$\lim_{n\to\infty}\left(-\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{n}\right) = -\frac{1}{2} - \frac{1}{2}\cdot 0 = -\frac{1}{2}.$$

(b) Have a look at the function to try to get an idea of it before just starting to compute. Dividing top and bottom by n, we have

$$\frac{3n^2 + n^3}{2n^2} = \frac{3 + n}{2} = \frac{3}{2} + \frac{n}{2},$$

which is certainly not bounded above. Hence our sequence $1 - (3n^2 + n^3)/(2n^2)$ is not bounded below. Thus, by Theorem 2.3.9 it is also not convergent.

(c) As we did on the previous sheet, we compute

$$\sqrt{n^2+1}-n=\frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n}=\frac{((n^2+1)-n^2)}{\sqrt{n^2+1}+n}=\frac{1}{\sqrt{n^2+1}+n}.$$

Next, we want to use the Sandwich Theorem, which means we can get a "nicer" upper bound. Since $\sqrt{n^2+1}+n>\sqrt{n^2}+n=2n$, we get

$$0<\sqrt{n^2+1}-n=\frac{1}{\sqrt{n^2+1}+n}<\frac{1}{2n}.$$

Finally, as we have seen before, $(b_n)_{n\geq 1}=(1/2n)_{n\geq 1}$ is a null sequence, and so by the Sandwich Theorem, so is $(\sqrt{n^2+1}-n)_{n\geq 1}$.

(d) This again looks like the sequence $\sqrt{2+n}-\sqrt{n}$, but the behaviour is very different. Indeed, in this case we can manipulate it to:

$$\sqrt{2n}-\sqrt{n} \;=\; \sqrt{2}\sqrt{n}-\sqrt{n} \;=\; \big(\sqrt{2}-1\big)\sqrt{n}.$$

Once again this is "clearly" not bounded (and it is fine if you finish the argument with that comment). But if you want to go on to prove it more carefully: suppose, for a contradiction, that the sequence is bounded above by, say, ℓ. Then, for all n we have that $(\sqrt{2}-1)\sqrt{n}\leq\ell$ which, after manipulation, gives

$$n\leq\frac{\ell^2}{(\sqrt{2}-1)^2}\qquad\text{for all } n.$$

Since $\mathbb{N}$ is unbounded this is a contradiction.

(e) Maybe you noticed that $0<3^{-n}<2^{-n}$. By Lemma 3.1.6 (or the solution to Exercise 2(e) on the Week 3 sheet), $\lim_{n\to\infty}(2^{-n})=0$. Thus by the Sandwich Theorem $\lim_{n\to\infty}(3^{-n})=0$.

(f) This is like Example 3.1.8. Using 3.1.6,

$$0<\frac{n^3}{3^n+4^n}\leq\frac{n^3}{4^n}=\frac{n^3}{2^n2^n}\leq\frac{n^3}{n^2n^2}=\frac{1}{n}.$$

As usual, $\lim_{n\to\infty}1/n=0$ and so by the Sandwich Theorem our given sequence has limit 0.

Question 2: Let $(a_n)_{n\geq 1}$ be a bounded, decreasing sequence. Prove that $(a_n)_{n\geq 1}$ is convergent.

Solution: This is the natural variant of Theorem 2.5.3 (which considered increasing sequences). There are two obvious proofs. First, you could take the proof of that theorem and replace $\geq$ by $\leq$ and supremum by infimum in all the appropriate places. Or, you could take a new sequence $\{b_n=-a_n\}$ and apply Theorem 2.5.3 to that. Either is fine.

First Proof: Since the set $S=\{a_n : n\in\mathbb{N}\}$ is bounded, it has an infimum, ℓ say, by the notes (explicitly Theorem 2.4.11). We will show that $\ell=\lim_{n\to\infty}a_n$.

So let $\varepsilon>0$ be given. Arguing as in Lemma 13.2.6 there exists $x\in\{a_n : n\in\mathbb{N}\}$ such that $\ell-\varepsilon<x<\ell+\varepsilon$. Consequently $\ell<x<\ell+\varepsilon$ since ℓ is a lower bound for S. Note that $x=a_N$ for some N and hence $\ell<a_N<\ell+\varepsilon$.

For any $n \geq N$ we have that $a_n \leq a_N$ (since the sequence $(a_n)_{n\geq 1}$ is decreasing) and $a_n \geq \ell$ (since ℓ is a lower bound for S). Thus, for all $n \geq N$, $\ell < a_n < \ell + \varepsilon$, as required.

Second Proof: So, we are given a sequence $(a_n)_{n\geq 1}$ that is both decreasing and bounded below; thus $a_n \leq a_{n-1}$ for all n and there exists ℓ with $a_n \geq \ell$ for all n. Now set $b_n = -a_n$. Then these two hypotheses mean that $b_n = -a_n \geq -a_{n-1} = b_{n-1}$ and $b_n = -a_n \leq -\ell$. In other words, $(b_n)_{n\geq 1}$ is a bounded above, increasing sequence.

By Theorem 2.5.3 it therefore has a limit, say m and, for all $\varepsilon > 0$ there exists N such that $m - \varepsilon < b_n \leq m$, for all $n \geq N$. Taking negatives we get

$$\text{for all } \varepsilon > 0 \text{ there exists } N \text{ such that } -m + \varepsilon > -b_n \geq -m, \text{ for all } n \geq N.$$

In other words, $(a_n)_{n\geq 1} = (-b_n)_{n\geq 1}$ has limit $-m$.

Remark. An important point here is that:

(a) *If $(a_n)_{n\geq 1}$ is a bounded increasing sequence then, by Theorem 2.5.3 it has a limit ℓ. This ℓ satisfies $\ell \geq a_n$ for all n.*

Proof: Just notice that ℓ was defined to be the supremum of $\{a_n\}$ and so must satisfy $\ell \geq a_n$ for each n.

(b) *Similarly if $(a_n)_{n\geq 1}$ is a bounded decreasing sequence then, by Question 2 it has a limit ℓ. This ℓ satisfies $\ell \leq a_n$ for all n.*

Proof: Similar.

Question 3: a. Define the sequence $(a_n)_{n\geq 1}$ inductively by $a_1 = 1$ and $a_{n+1} = a_n/3 + 1$.

i. Prove that $a_n \leq 3/2$ for all $n \geq 1$,

ii. Prove that $(a_n)_{n\geq 1}$ is an increasing sequence.

iii. What Theorem implies the sequence converges? Show that the limit is 4/3.

b. What happens if the starting value is $a_1 = 4$?

Solution

i. Proof by induction. The base case is true, $a_1 = 1 \leq 3/2$.

Assume true for $n = k$, so $a_k \leq 3/2$. Consider

$$a_{k+1} = \frac{a_k}{3} + 1 \leq \frac{3}{6} + 1 = \frac{3}{2}.$$

Hence the result holds for $n = k + 1$ and thus, by induction, for all $n \geq 1$.

ii. By induction show that $a_{n+1} - a_n > 0$ for all $n \geq 1$. For $n = 1$ this follows since $a_2 = 4/3 > 1 = a_1$. Assume the result is true for $n = k$, so $a_{k+1} - a_k \geq 0$. Consider

$$a_{k+2} - a_{k+1} = (a_{k+1}/3 + 1) - (a_k/3 + 1) = (a_{k+1} - a_k)/3 > 0,$$

by the inductive hypothesis. Hence the result is true for $n = k + 1$ and so, by induction, for all $n \geq 1$. Thus $(a_n)_{n\geq 1}$ is an increasing sequence.

iii. The Monotone Convergence Theorem implies the sequence has a limit, ℓ say. Then, by the Algebra of Limits,

$$a_{n+1} = a_n/3 + 1 \to \ell/3 + 1$$

as $n \to \infty$. That is, the sequence $(a_{n+1})_{n\geq 1}$ converges to $\ell/3 + 1$. Yet $(a_{n+1})_{n\geq 1}$ is the subsequence of $(a_n)_{n\geq 1}$ obtained by omitting the first term and so, by a result in the course, has the same limit ℓ. Therefore $\ell = \ell/3 + 1$ which gives $\ell = 3/2$.

Question 4 What is

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}?$$

Hint Consider it the limit (if it exists) of

$$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},\ \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}},\ldots$$

i.e 1.414..., 1.8477..., 1.961..., 1.990..., 1.997..., 1.999...

i. The evidence here is that the sequence is increasing. Prove it is.

ii. Is it bounded above? Prove that it is.

iii. What Theorem implies the sequence converges? Show that the limit is 2.

Further Hint Find an iterative definition of the sequence.

Solution Define the sequence by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2+a_n}$ for all $n \geq 1$.
i. Proof by induction that $a_{n+1} \geq a_n$ for all $n \geq 1$.

Base case, $n = 1$, follows from $a_2 = \sqrt{2+\sqrt{2}} \geq \sqrt{2+0} = a_1$.

Assume true for $n = k$ so $a_{k+1} \geq a_k$. Consider

$$a_{k+2}^2 - a_{k+1}^2 = (2+a_{k+1}) - (2+a_k) = a_{k+1} - a_k \geq 0$$

by the inductive hypothesis. Hence $a_{k+2}^2 \geq a_{k+1}^2$ and thus, since all terms are positive, $a_{k+2} \geq a_{k+1}$. So true for $n = k+1$ and thus, by induction, for all $n \geq 1$.

ii. You have to guess a possible upper bound and then prove it is. From the (limited) evidence we might guess 2. Certainly $a_1 = \sqrt{2} < 2$ and if $a_k < 2$ then $a_{k+1} = \sqrt{2+a_k} < \sqrt{2+2} = 2$. Thus, by induction, the sequence is bounded above by 2.

iii By the Monotone Convergence Theorem the sequence converges. Let $\ell = \lim_{n\to\infty} a_n = \sqrt{2+l}$. By results in the course we have

$$a_{n+1} = \sqrt{2+a_n} \to \sqrt{2+\ell}$$

as $n \to \infty$. That is, the sequence $(a_{n+1})_{n\geq 1}$ converges to $\sqrt{2+\ell}$. Yet $(a_{n+1})_{n\geq 1}$ is the subsequence of $(a_n)_{n\geq 1}$ obtained by omitting the first term and so, by a result in the course, has the same limit ℓ. That is

$$\ell = \lim_{n\to\infty} a_{n+1} = \sqrt{2+\ell}.$$

The solutions of $\ell^2 - \ell - 2 = 0$ are $\ell = 2$ and -1. Since the sequence is increasing and started at 1 the limit must be positive, i.e. $\ell = 2$.

Extra Question for Week 4: Suppose that $p(x)$ and $q(x)$ are polynomials with real coefficients, and $q(x) \neq 0$. What is the limit of the sequence $a_n = p(n)/q(n)$ as $n \to \infty$?

Solution: First, let's get rid of a small point: there could be integer values n such that $q(n) = 0$, so then a_n is not defined. So either disqualify all q with a positive integer solution. Alternatively, use the fact that q can have only finitely many solutions so, for large enough n, we do have a_n well-defined, so just ignore the finitely many undefined values (after all, finitely many terms make no difference to the limit of a sequence).

Next, "it depends" is a correct answer, but a rather lazy one; surely you can do better than that! So look at some particular examples to get an idea of what can happen. You will probably see that if the degree of p is strictly bigger than that of q then the sequence is not convergent, if the degrees of p and q are equal then the sequence converges to a nonzero value (the leading coefficient (l.c.) of p divided by the leading coefficient of q), and if the degree of p is strictly smaller than that of q then the sequence converges to 0.

Once you see that, it's kind of obvious, but can you prove it? You might come up with a proof based on the procedure of dividing top and bottom by x^d where d is the degree of q, let's write $d = \deg(q(x))$ for degree.

You could (but don't have to) use the division theorem for polynomials: divide $p(x)$ by $q(x)$ to get a quotient and remainder: $p(x) = q(x)a(x) + r(x)$ where $a(x)$ and $r(x)$ are polynomials and the degree of $r(x) = 0$ is strictly smaller than the degree of $q(x)$. Therefore

$$a_n = \frac{p(n)}{q(n)} = a(n) + \frac{r(n)}{q(n)}.$$

Then divide into the cases:

$\deg(p(x)) < \deg(q(x)))$, so $a(x) = 0$;

$\deg(p(x)) = \deg(q(x)))$, so $a(x)$ is a constant (the l.c. of p divided by the l.c. of q)

$\deg(p(x)) > \deg(q(x))$, so $a(x)$ is a polynomial of degree > 1, hence $a_n \to \infty$ as $n \to \infty$.

If we decide to go for a proper proof that the three cases converge/don't converge as stated above, then we can argue as follows.

Claim: if $p(x), q(x)$ are polynomials with $\deg p(x) < \deg q(x)$ then

$$\frac{p(n)}{q(n)} \to 0$$

as $n \to \infty$.

Proof: Say $p(x) = \sum_{i=0}^{m} a_i x^i$, $q(x) = \sum_{j=0}^{k} b_i x^i$, with the $a_i, b_i \in \mathbb{R}$, $a_m \neq 0$, $b_n \neq 0$ and $\deg p(x) = m < k = \deg q(x)$. Consider $p(x)/q(x)$ and divide top and bottom by x^k, to get

$$\frac{p(n)}{q(n)} = \frac{\sum_{i=0}^{m} a_i x^{i-k}}{\sum_{j=0}^{k} b_i x^{i-k}} \qquad (*).$$

For the rest of the argument we substitute the integer variable n for x. Note that every term in this expression is a negative power of n except the leading term on the bottom line, which is b_0. By the Algebra of Limits Theorem applied (a number of times) to $(*)$ and the already-proved-in-the-notes/easy-to-prove fact that $n^j \to 0$ as $n \to \infty$ whenever j is a negative integer, we deduce that the limit of $p(n)/q(n)$ as $n \to \infty$ is $0/b_0 = 0$. As claimed.

The other two cases are dealt with similarly or, note that the case we've just done shows that the "fractional part" $r(n)/q(n)$, of the quotient $r(n)/q(n)$ has limit 0, so you can concentrate on the other term, $a(n)$, which is either a constant (which will therefore be the limit) or a nonconstant polynomial (which is therefore unbounded, hence the sequence is not convergent).