Degrees of Guaranteed Envy-Freeness in Finite Bounded Cake-Cutting Protocols

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Abstract. Fair allocation of goods or resources among various agents is a central task in multiagent systems and other fields. The specific setting where just one divisible resource is to be divided fairly is commonly referred to as cake-cutting, and agents are called players in this setting. Cake-cutting protocols aim at dividing a cake and assigning the resulting portions to several players in a way that each of the players, according to his or her valuation of these portions, feels to have received a "fair" amount of the cake. An important notion of fairness is envy-freeness: No player wishes to switch the portion of the cake received with another player's portion. Despite intense efforts in the past, it is still an open question whether there is a *finite bounded* envy-free cake-cutting protocol for an arbitrary number of players, and even for four players. In this paper, we introduce the notion of degree of guaranteed envy-freeness (DGEF, for short) as a measure of how good a cake-cutting protocol can approximate the ideal of envy-freeness while keeping the protocol finite bounded. We propose a new finite bounded proportional protocol for any number $n \geq 3$ of players, and show that this protocol has a DGEF of $1 + \lceil n^2/2 \rceil$. This is the currently best DGEF among known finite bounded cake-cutting protocols for an arbitrary number of players. We will make the case that improving the DGEF even further is a tough challenge, and determine, for comparison, the DGEF of selected known finite bounded cake-cutting protocols, among which the Last Diminisher protocol turned out to have the best DGEF, namely, $2 + n(n-1)/2$. Thus, the Last Diminisher protocol has $\lceil n/2 \rceil - 1$ fewer guaranteed envy-free-relations than our protocol.

Keywords: cake-cutting protocol, fair division, multiagent resource allocation

1 Introduction

Research in the area of cake-cutting started off in the 1940s with the pioneering work of Steinhaus [18] who, to the best of our knowledge, was the first to introduce the problem of fair division. Dividing a good (or a resource) fairly among several players such that each of them is satisfied with the portion received is of central importance in many fields. In the last 60 years this research area has developed vividly, spreading out into various directions and with applications in areas as diverse as economics, mathematics, computer science, and psychology. While some lines of this research seek to find reasonable interpretations of what "fairness" really stands for and how to measure it [9, 7], others study proofs of existence or impossibility theorems regarding fair division (see, e.g., [1]), or design new cake-cutting procedures [3, 21, 17] and, relatedly, analyze their complexity with respect to both upper and lower bounds [14, 23, 15]. As cake-cutting procedures involve several players, they are also referred to as "protocols."

Cake-cutting protocols aim at achieving a *fair* division of an infinitely divisible resource among n players, who each may have different valuations of different parts of the resource. We focus on the notion of envy-freeness in finite bounded cake-cutting protocols. Cake-cutting protocols are either finite or continuous. While a *finite* protocol always provides a solution after only a finite number of decisions, a *continuous* protocol could potentially run forever. Among finite protocols, one can further distinguish between bounded and unbounded ones. A finite *bounded* cake-cutting protocol is present if we know in advance that a certain number of steps (that may depend on the number of players) will suffice to divide the resource fairly—independently of how the players may value distinct parts of the resource in a particular case and independently of the strategies chosen by the players. In contrast, in finite *unbounded* cake-cutting protocols, we cannot predict an upper bound on how many steps will be required to achieve the same goal. Aiming to apply cake-cutting procedures to real-world scenarios, it is important to develop fair *finite bounded* cake-cutting protocols. In this context, "fairness" is often interpreted as meaning "envy-freeness." A division is *envy-free* if no player has an incentive to switch his or her portion with the portion any other player received.

For the division of a divisible good among n players, Steinhaus [19] proved that an envy-free division always exists. However, the current state of the art—after six decades of intense research—is that for arbitrary n, and even for $n = 4$, the development of *finite bounded* envy-free cake-cutting protocols still appears to be out of reach, and a big challenge for future research. For $n > 3$ players, hardly any envy-free cake-cutting protocol is known, and the ones that are known are either finite unbounded or continuous (see, e.g., [3, 16, 5]).

Our goal in this paper is to look for compromises that can be made with respect to envy-freeness while keeping the protocol finite bounded: We propose an approach to evaluate finite bounded (yet possibly non-envy-free) cake-cutting protocols with respect to their *degree of guaranteed envy-freeness* (DGEF), a notion to be formally introduced in Section 3. Informally put, this notion provides a measure of how good such a protocol can approximate the (possibly for this particular protocol unreachable) ideal of envy-freeness in terms of the number of envy-free-relations that exist even in the worst case. To put the DGEF approach into practice, we present a new finite bounded proportional cake-cutting protocol with a significantly enhanced degree of guaranteed envy-freeness in

Section 4, and discuss its significance in Section 5. A comparison to related work is drawn in the full version of this paper [13].

2 Preliminaries and Notation

Cake-cutting is about dividing a cake into portions that are assigned to the players such that each of them feels, according to his or her valuation of the portions, to have received a fair amount of the cake (where "cake" is a metaphor for the resource or the good to be divided). The cake is assumed to be infinitely divisible and can be divided into arbitrary pieces without losing any of its value. Given n players, cake C is to be divided into n portions that are to be distributed among the players so as to satisfy each of them. A portion is not necessarily a single piece of cake; it can be a collection of disjoint, possibly noncontiguous pieces of C. The players may have different individual valuations of the single pieces of the cake: One player may prefer the pieces with the chocolate on top, whereas another player may prefer the pieces with the cherry topping.

More formally, cake C is represented by the unit interval $[0,1]$ of real numbers. By performing cuts, C is divided into m pieces c_k, $1 \leq k \leq m$, which are represented by subintervals of $[0,1]$. Each player p_i, $1 \leq i \leq n$, assigns value $v_i(c_k) = v_i(x_k, y_k)$ to piece $c_k \subseteq C$, where c_k is represented by the subinterval $[x_k, y_k] \subseteq [0,1]$ and p_i's valuation function v_i maps subintervals of $[0,1]$ to real numbers in $[0,1]$. We require each function v_i to satisfy the following properties:

1. *Normalization:* $v_i(0,1) = 1$.
2. *Positivity:*[1] For all $c_k \subseteq C$ with $c_k \neq \emptyset$ we have $v_i(c_k) > 0$.
3. *Additivity:* For all $c_k, c_\ell \subseteq C$ with $c_k \cap c_\ell = \emptyset$ we have $v_i(c_k) + v_i(c_\ell) = v_i(c_k \cup c_\ell)$.
4. *Divisibility:*[2] For all $c_k \subseteq C$ and for each α, $0 \leq \alpha \leq 1$, there exists some $c_\ell \subseteq c_k$ such that $v_i(c_\ell) = \alpha \cdot v_i(c_k)$.

For simplicity, we write $v_i(x_k, y_k)$ instead of $v_i([x_k, y_k])$ for intervals $[x_k, y_k] \subseteq [0,1]$. Due to Footnote 2, no ambiguity can arise. For each $[x,y] \subseteq [0,1]$, define $\|[x,y]\| = y - x$.

We assume C to be heterogeneous (i.e., subintervals of $[0,1]$ having equal size can be valued differently by the same player). Moreover, distinct players may value one and the same piece of C differently, i.e., their individual valuation functions will in general be distinct. Every player knows only the value of

[1] The literature is a bit ambiguous regarding this assumption. Some papers require the players' values for nonempty pieces of cake to be *nonnegative* (i.e., $v_i(c_k) \geq 0$) instead of positive.

[2] Divisibility implies that for each $x \in [0,1]$, $v_i(x,x) = 0$. That is, isolated points are valued 0, and open intervals have the same value as the corresponding closed intervals.

(arbitrary) pieces of C corresponding to his or her own valuation function. Players do not have any knowledge about the valuation functions of other players.

A *division of* C is an assignment of disjoint and nonempty portions $C_i \subseteq C$, where $C = \bigcup_{i=1}^{n} C_i = \bigcup_{k=1}^{m} c_k$, to the players such that each player p_i receives a portion $C_i \subseteq C$ consisting of at least one nonempty piece $c_k \subseteq C$. The goal is to assign all portions in as fair a way as possible. There are different interpretations, though, of what "fair" might mean. To distinguish between different degrees of fairness, among others, the following two notions have been introduced in the literature (see, e.g., [17]):

Definition 1. *Let $v_1, v_2, \ldots, v_n$ be the valuation functions of the n players. A division of cake $C = \bigcup_{i=1}^{n} C_i$, where C_i is the ith player's portion, is said to be: (i)* simple fair *(a.k.a.* proportional*) if for each i, $1 \leq i \leq n$, $v_i(C_i) \geq 1/n$; (ii)* envy-free *if for each i and j, $1 \leq i, j \leq n$, $v_i(C_i) \geq v_i(C_j)$.*

A cake-cutting protocol describes an interactive procedure for obtaining a division of a given cake. A protocol is characterized by a set of rules and a set of strategies (see, e.g., [4]), which have to be followed by all players for them to be guaranteed a fair portion of the cake. The rules determine the course of action, such as a request to cut the cake, whereas the strategies define how to achieve a certain degree of fairness, e.g., by advising the players where to cut the cake.

3 Degrees of Guaranteed Envy-Freeness

The design of envy-free cake-cutting protocols for any number n of players seems to be quite a challenge. For $n \leq 3$ players, several protocols that always provide envy-free divisions have been published, both finite (bounded and unbounded) and continuous ones [21, 4, 17]. However, to the best of our knowledge, up to date no finite bounded cake-cutting protocol for $n > 3$ players is known to always provide an envy-free division. For practical purposes, it would be most desirable to have *finite bounded* cake-cutting protocols that always provide divisions as fair as possible. We propose an approach that weakens the concept of envy-freeness for the purpose of keeping the protocols finite bounded.

On the one hand, in this section we study known simple fair (i.e., proportional) cake-cutting protocols that are finite bounded, and determine their "degree of guaranteed envy-freeness" (see Definition 3). On the other hand, in Section 4 we propose a new finite bounded proportional cake-cutting protocol that—compared with the known protocols—has an enhanced DGEF.

When investigating the degree of envy-freeness of a cake-cutting protocol for n players, for each player p_i, $1 \leq i \leq n$, the value of his or her portion needs to be compared to the values of the $n-1$ other portions (according to the measure of player p_i).[3] Thus, $n(n-1)$ pairwise relations need to be investigated in order to determine the degree of envy-freeness of a cake-cutting protocol for n players. A player p_i envies another player p_j, $1 \leq i, j \leq n$, $i \neq j$, when p_i prefers player p_j's portion to his or her own. If p_i envies p_j, we call the relation between these two players an *envy-relation*; otherwise, we call it an *envy-free-relation*.

Definition 2. Let cake $C = \bigcup_{i=1}^{n} C_i$ be divided among all players in $P = \{p_1, p_2, \ldots, p_n\}$, where v_i is p_i's valuation function and C_i is p_i's portion. Let $p_i, p_j \in P$ be any two distinct players. An *envy-relation* occurs in this division if p_i envies p_j (denoted by $p_i \Vdash p_j$), i.e., if $v_i(C_i) < v_i(C_j)$; an *envy-free-relation* occurs if p_i does not envy p_j (denoted by $p_i \nVdash p_j$), i.e., if $v_i(C_i) \geq v_i(C_j)$.

We mention the following properties of envy-relations and envy-free-relations.[4] No player can envy him- or herself, i.e., envy-relations are irreflexive: The inequality $v_i(C_i) < v_i(C_i)$ never holds. Thus, $v_i(C_i) \geq v_i(C_i)$ always holds. However, when counting envy-free-relations for a given division, we disregard the trivial envy-free-relations $p_i \nVdash p_i$, $1 \leq i \leq n$. Neither envy-relations nor envy-free-relations need to be transitive. These observations imply that envy-relations and envy-free-relations are either one-way or two-way, i.e., it is possible that: (a) two players envy each other ($p_i \Vdash p_j$ and $p_j \Vdash p_i$), (b) neither of two players envies the other ($p_i \nVdash p_j$ and $p_j \nVdash p_i$), (c) one player envies another player but is not envied by this other player ($p_i \Vdash p_j$ and $p_j \nVdash p_i$).

Assuming all players to follow the rules and strategies, some cake-cutting protocols always guarantee an envy-free division (i.e., they always find an envy-free division of the cake), whereas others do not. Only protocols that *guarantee* an envy-free division in *every* case, even in the worst case (in terms of the players' valuation functions), are considered to be envy-free. An envy-free division may be obtained by coincidence, just because the players have matching valuation functions that avoid envy, and not because envy-freeness is enforced by the rules and strategies of the cake-cutting protocol used. In the worst case, however, when the players have totally nonconforming valuation functions, an envy-free division would not just happen by coincidence, but needs to be en-

[3] We will use "valuation" and "measure" interchangeably.

[4] Various analogs of envy-relations and envy-free-relations have also been studied, from an economic perspective, in the different context of multiagent allocation of indivisible resources. Feldman and Weiman [10] consider "non-envy relations," which are similar to our notion of envy-free-relations, and Chauduri [6] introduces "envy-relations." Despite some similarities, their notions differ from ours, both in their properties and in the way properties holding for their and our notions are proven. For example, Chauduri [6] notes that mutual envy cannot occur in a market equilibrium, i.e., in this case his "envy-relations" are asymmetric, which is in sharp contrast to two-way envy being allowed for our notion.

forced by the rules and strategies of the protocol. An envy-free-relation is said to be *guaranteed* if it exists even in the worst case.

Definition 3. For $n \geq 1$ players, the *degree of guaranteed envy-freeness* (DGEF, for short) of a given proportional[5] cake-cutting protocol is defined to be the maximum number of envy-free-relations that exist in every division obtained by this protocol (provided that all players follow the rules and strategies of the protocol), i.e., the DGEF (which is expressed as a function of n) is the number of envy-free-relations that can be guaranteed even in the worst case.

This definition is based on the idea of weakening the notion of fairness in terms of envy-freeness in order to obtain cake-cutting protocols that are fair (though perhaps not envy-free) *and* finite bounded, where the fairness of a protocol is given by its degree of guaranteed envy-freeness. The higher the degree of guaranteed envy-freeness the fairer the protocol.

Proposition 1 gives an upper and a lower bound on the degree of guaranteed envy-freeness for proportional cake-cutting protocols. Its proof can be found in the full version of this paper [13].

Proposition 1. *Let $d(n)$ be the degree of guaranteed envy-freeness of a proportional cake-cutting protocol for $n \geq 2$ players. It holds that $n \leq d(n) \leq n(n-1)$.*

An envy-free cake-cutting protocol for n players guarantees that no player p_i envies any other player p_j, i.e., the DGEF of an envy-free protocol equals $n(n-1)$, the upper bound in Proposition 1.

The degree of fairness of a division obtained by applying a proportional cake-cutting protocol highly depends on the rules of this protocol. Specifying and committing to appropriate rules often increases the degree of guaranteed envy-freeness, whereas the lack of such rules jeopardizes it in the sense that the number of guaranteed envy-free-relations may be limited to the worst-case minimum of n as stated in Proposition 1. In this context, "appropriate rules" are those that involve the players' evaluations of other players' pieces and portions that still are to be assigned. Concerning a particular piece of cake, involving the evaluation of as many players as possible in the allocation process helps to keep the number of envy-relations to be created low, since this allows to determine early on whether a planned allocation later may turn out to be disadvantageous—and thus allows to take adequate countermeasures. In contrast, omitting mutual evaluations means to forego additional knowledge that could turn out to be most valuable later on. For example, say player p_i is going to get assigned piece c_j.

[5] We restrict the notion of DGEF to proportional protocols only, since otherwise the DGEF may overstate the actual level of fairness, e.g., if all the cake is given to a single player.

If the protocol asks all other players to evaluate piece c_j according to their measures, all envy-relations to be created by the assignment of piece c_j to player p_i can be identified before the actual assignment and thus countermeasures (such as trimming piece c_j) can be undertaken. However, if the protocol requires no evaluations on behalf of the other players, such potential envy-relations cannot be identified early enough to prevent them from happening.

Lemma 1. *A proportional cake-cutting protocol with $n \geq 2$ players has a DGEF of n (i.e., each player is guaranteed only one envy-free-relation) if the rules of the protocol require none of the players to value any of the other players' portions.*

Our next result shows the DGEF for a number of well-known finite bounded *proportional* cake-cutting protocols. Note that these protocols have been developed with a focus on achieving proportionality, and not on maximizing the DGEF. The proofs of Lemma 1 and Theorem 1 can be found in the full version of this paper [13].

Theorem 1. *For $n \geq 3$ players, the proportional protocols in Table 1 have a DGEF as shown in the same table.*

Table 1. DGEF of selected finite bounded cake-cutting protocols.

Protocol	DGEF
Last Diminisher [18]	$2 + n(n-1)/2$
Lone Chooser [11]	n
Lone Divider [12]	$2n-2$
Cut Your Own Piece (no strategy) [20]	n
Cut Your Own Piece (left-right strategy)	$2n-2$
Divide and Conquer [8]	$n \cdot \lfloor \log n \rfloor + 2n - 2^{\lfloor \log n \rfloor + 1}$
Minimal-Envy Divide and Conquer [2]	$n \cdot \lfloor \log n \rfloor + 2n - 2^{\lfloor \log n \rfloor + 1}$
Recursive Divide and Choose [22]	n

4 A Protocol with an Enhanced DGEF

Figure 1 shows a finite bounded proportional cake-cutting protocol with an enhanced DGEF for n players, where $n \geq 3$ is arbitrary. Unless specified otherwise, ties in this protocol can be broken arbitrarily. Regarding the DGEF results in Table 1 the Last Diminisher protocol[6] shows the best results for $n \geq 6$, whereas

[6] This protocol has been developed by Banach and Knaster and was first presented in Steinhaus [18].

the best results for $n < 6$ are achieved by the Last Diminisher protocol as well as both the Divide and Conquer protocols [8, 2]. The protocol in Figure 1 improves upon these degrees of guaranteed envy-freeness for all $n \geq 3$ and improves upon the DGEF of the Last Diminisher protocol by $\lceil n/2 \rceil - 1$ additional guaranteed envy-free-relations.

Before presenting our protocol in detail, let us give an intuitive, high-level explanation. Both the protocol in Figure 1 and the Last Diminisher protocol are, more or less, based on the same idea of determining a piece of minimal size that is valued exactly $1/n$ by one of the players (who is still in the game), which guarantees that all other players (who are still in the game) will not envy this player for receiving this particular piece. However, the protocol in Figure 1 works in a more parallel way, which makes its enhanced DGEF of $\lceil n^2/2 \rceil + 1$ possible (see Theorem 3). To ensure that the parallelization indeed pays off in terms of increasing the degree of guaranteed envy-freeness, the "inner loop" (Steps 4.1 through 4.3) of the protocol is decisive. In addition, the protocol in Figure 1 provides a proportional division in a finite bounded number of steps (see Theorem 2), just as the Last Diminisher protocol.

Remark 1. Some remarks on the protocol in Figure 1 are in order:

1. From a very high-level perspective the procedure is as follows: The protocol runs over several rounds in each of which it is to find a player p_j who takes a portion from the left side of the cake, and to find a player p_k who takes a disjoint portion from the right side of the cake, such that none of the players still in the game envy p_j or p_k (at this, appropriate "inner-loop handling" might be necessary, see Figure 1 for details). Thereafter, p_j and p_k are to drop out with their portions, and a new round is started with the remaining cake (which is being renormalized, see remarks 3 and 4 below) and the remaining players. Finally, the Selfridge–Conway protocol is applied to the last three players in the game.[7]
2. The trivial cases $n = 1$ (where one player receives all the cake) and $n = 2$ (where each proportional division is always envy-free) are ignored.
3. Regarding $n \geq 5$ players, if at any stage of our protocol the same player marks both the leftmost smallest piece and the rightmost smallest piece, the cake may be split up into two pieces and later on merged again. To simplify matters, in such a case the interval boundaries are adapted as well, which is expressed in Step 8 of Figure 1. Simply put, the two parts of the cake are set next to each other again to ensure a seamless transition. This can be

[7] This protocol is known to be a finite bounded envy-free cake-cutting protocol for $n = 3$ players (see Stromquist [21]).

done without any loss in value due to additivity of the players' valuation functions.

4. In Steps 1 and 9.1, the value of subcake $C' \subseteq C$ is normalized such that $v_i(C') = 1$ for each player p_i, $1 \leq i \leq s$, for the sake of convenience. In more detail, each player p_i values C' at least s/n of C, i.e., $v_i(C') \geq (s/n) \cdot v_i(C)$. Thus, by receiving a proportional share (valued $1/s$) of C' each player p_i is guaranteed at least a proportional share (valued $1/n$) of C.

Input:	n players $p_1, p_2, \ldots, p_n$, where p_i has the valuation function v_i, and cake C.
Output:	Mapping of portions C_i to players p_i, where $C = \bigcup_{i=1}^{n} C_i$.
Initialization:	Set $\lambda := 0$, $\rho := 1$, $\rho' := \rho$, $s := n$, and $C' := [0,1] = C$.
While there are more than four players (i.e., $s > 4$), perform the outer loop (Steps 1 through 8).	
Step 1.	Let players p_i, $1 \leq i \leq s$, each make two marks at λ_i and ρ_i with $\lambda_i, \rho_i \in C'$ such that $v_i(\lambda, \lambda_i) = 1/s$ and $v_i(\rho_i, \rho) = 1/s$; note that $v_i(C') = 1$ (see Remark 1.4).
Step 2.	Find any player p_j such that there is no player p_z, $1 \leq j,z \leq s$, $j \neq z$, with $\|[\lambda, \lambda_z]\| < \|[\lambda, \lambda_j]\|$.
Step 3.	Find any player p_k such that there is no player p_z, $1 \leq k,z \leq s$, $k \neq z$, with $\|[\rho_z, \rho]\| < \|[\rho_k, \rho]\|$. If more than one player fulfills this condition for p_k, and p_j is one of them, choose p_k other than p_j.
If $j \neq k$, go directly to Step 5, else repeat the inner loop (Steps 4.1 through 4.3) until p_j and p_k are found with $j \neq k$, where p_j marks the leftmost smallest piece and p_k the rightmost smallest piece.	
Step 4.1.	Set $\rho' := \rho_k$.
Step 4.2.	Let players p_i, $1 \leq i \leq s$, each make a mark at $\rho_i \in C'$ such that $v_i(\rho_i, \rho') = 1/s$.
Step 4.3.	Find player p_k such that there is no player p_z, $1 \leq k,z \leq s$, $k \neq z$, with $\|[\rho_z, \rho']\| < \|[\rho_k, \rho']\|$. If more than one player fulfills this condition for p_k, and p_j is one of them, choose p_k other than p_j.
Step 5.	Assign portion $C_j = [\lambda, \lambda_j]$ to player p_j.
Step 6.	If $\rho = \rho'$, assign portion $C_k = [\rho_k, \rho]$ to player p_k, else assign portion $C_k = [\rho_k, \rho']$ to player p_k.
Step 7.	Let players p_j and p_k drop out.
Step 8.	If $\rho = \rho'$, set $C' := [\lambda_j, \rho_k]$ and $\rho := \rho_k$, else set $C' := [\lambda_j, \rho_k] \cup [\rho', \rho]$ and $\rho := \rho - \rho' + \rho_k$ (see Remark 1.3). Set $\lambda := \lambda_j$, $\rho' := \rho$, and $s := s - 2$.
Perform Steps 9.1 through 9.4 if and only if there are four players (i.e., $s = 4$). If there are three players (i.e., $s = 3$), go directly to Step 10.	
Step 9.1.	Let each p_i, $1 \leq i \leq s = 4$, make a mark at $\rho_i \in C'$ such that $v_i(\rho_i, \rho) = 1/s = 1/4$; note that $v_i(C') = 1$ (see Remark 1.4).
Step 9.2.	Find any player p_j such that there is no player p_k, $1 \leq j,k \leq s$, $j \neq k$ with $\|[\rho_k, \rho]\| < \|[\rho_j, \rho]\|$.
Step 9.3.	Assign portion $C_j = [\rho_j, \rho]$ to player p_j. Let player p_j drop out.
Step 9.4.	Set $\rho := \rho_j$ and $C' := [\lambda, \rho]$. Set $s := s - 1$.
Step 10.	Divide the remaining cake C' among the $s = 3$ remaining players by the Selfridge–Conway protocol.

Fig. 1. A proportional protocol with an enhanced DGEF of $\lceil n^2/2 \rceil + 1$ for $n \geq 3$ players.

Theorem 2. *The protocol in Figure 1 is finite bounded and proportional.*

The proofs of Theorems 2 and 3 are given in the full version of this paper [13], but we mention that the protocol is bounded by $(7 \cdot \lceil (n-4)/2 \rceil) + \left(3 \cdot \sum_{i=1}^{\lceil (n-4)/2 \rceil} (n-2i)\right) + 4 + 9$ steps, which shows that it indeed is finite bounded. The proof of proportionality follows along the lines of the proof of Theorem 3 and in particular uses that each player is assigned a portion valued exactly $1/s$ of a subcake that he or she values to be worth at least s/n of the given cake, according to his or her measure.

Theorem 3. *For $n \geq 5$ players, the cake-cutting protocol in Figure 1 has a DGEF of $\lceil n^2/2 \rceil + 1$.*[8]

5 Discussion

It may be tempting to seek to decrease envy (and thus to increase the DGEF) via trading, aiming to get rid of potential circular envy-relations. Indeed, if the DGEF is *lower than* $n(n-1)/2$, the number of guaranteed envy-free-relations can be improved to this lower bound by resolving circular envy-relations (of which two-way envy-relations are a special case) by means of circular trades after the execution of the protocol.[9] Thus, in this case, involving subsequent trading actions adds on the number of guaranteed envy-free-relations. Furthermore, having $n(n-1)/2$ guaranteed envy-free-relations after all circular envy-relations have been resolved, three more guaranteed envy-free-relations can be gained by applying an envy-free protocol (e.g., the Selfridge–Conway protocol) to the three most envied players, which yields to an overall lower bound of $3 + n(n-1)/2$ guaranteed envy-free-relations. Note, though, that the DGEF is defined to make a statement on the performance of a particular protocol and not about all sorts of actions to be undertaken afterwards.

[8] Note that the same formula holds if $n = 3$, but for the special case of $n = 4$ (see [13] for details) even one more envy-free-relation can be guaranteed (i.e., for $n = 4$ players, the DGEF of the protocol in Figure 1 is $(n^2/2) + 2$).

[9] To be specific here, all occurrences of "guaranteed envy-free-relations" in this and the next paragraph refer to those envy-free-relations that are guaranteed to exist after executing some cake-cutting protocol *and in addition, subsequently, performing trades that are guaranteed to be feasible*. This is in contrast with what we mean by this term anywhere else in the paper; "guaranteed envy-free-relations" usually refers to those envy-free-relations that are guaranteed to exist after executing the protocol only. As is common, we consider trading not to be part of a cake-cutting protocol, though it might be useful in certain cases (for example, Brams and Taylor mention that trading might be used "to obtain better allocations; however, this is not a procedure but an informal adjustment mechanism" [4, page 44]). In particular, the notion of DGEF refers to (proportional) cake-cutting protocols without additional trading.

However, if the DGEF of a proportional cake-cutting protocol is $n(n-1)/2$ *or higher* (such as the DGEF of the protocol presented in Figure 1) then circular envy-relations are not *guaranteed* to exist, and hence, in this case, trading has no impact on the number of guaranteed envy-free-relations.

Although the well-known protocols listed in Table 1 have not been developed with a focus on maximizing the DGEF,[10] linking their degrees of guaranteed envy-freeness to the lower bound provided by involving, e.g., the Selfridge–Conway protocol and guaranteed trading opportunities indicates that the development of cake-cutting protocols with a considerably higher DGEF or even with a DGEF close to the maximum of $n(n-1)$ poses a true challenge. That is why we feel that the enhanced DGEF of the protocol presented in Figure 1 constitutes a significant improvement.

6 Conclusions

Finite bounded protocols that guarantee an envy-free division for $n > 3$ players are still a mystery. However, finite bounded protocols are the ones we are looking for in terms of practical implementations. We propose to weaken the requirement of envy-freeness, while insisting on finite boundedness. To this end, we introduced the notion of degree of guaranteed envy-freeness for proportional cake-cutting protocols and determined the DGEF in existing finite bounded proportional cake-cutting protocols. We expect that the concept of DGEF is suitable to extend the scope for the development of new finite bounded cake-cutting protocols by allowing to approximate envy-freeness step by step. In this context, we proposed a new finite bounded proportional cake-cutting protocol, which provides a significantly enhanced DGEF compared with those in Table 1. In particular, our protocol has $\lceil n/2 \rceil - 1$ more guaranteed envy-free-relations than the Last Diminisher protocol, which previously was the best finite bounded proportional protocol with respect to the DGEF. To achieve this significantly enhanced DGEF, our protocol makes use of parallelization with respect to the leftmost and the rightmost pieces. In this regard, adjusting the values of the pieces to be marked from $1/n$ to $1/s$ (with s players still in the game) and applying an appropriate inner-loop procedure is crucial to make the parallelization work.

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[10] Quite remarkably, without any trading actions and without involving, e.g., the Selfridge–Conway protocol the Last Diminisher protocol achieves with its DGEF almost (being off only by one) the trading- and Selfridge–Conway-related bound of $3 + n(n-1)/2$ mentioned above.

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References

1. E. Akin. Vilfredo Pareto cuts the cake. *Journal of Mathematical Economics*, 24(1):23–44, 1995.
2. S. Brams, M. Jones, and C. Klamler. Divide-and-Conquer: A proportional, minimal-envy cake-cutting procedure. In *Dagstuhl Seminar 07261*. Dagstuhl Seminar Proceedings, 2007.
3. S. Brams and A. Taylor. An envy-free cake division protocol. *The American Mathematical Monthly*, 102(1):9–18, 1995.
4. S. Brams and A. Taylor. *Fair Division: From Cake-Cutting to Dispute Resolution*. Cambridge University Press, 1996.
5. S. Brams, A. Taylor, and W. Zwicker. A moving-knife solution to the four-person envy-free cake-division problem. *Proceedings of the American Mathematical Society*, 125(2):547–554, 1997.
6. A. Chauduri. Formal properties of interpersonal envy. *Theory and Decision*, 18:301–312, 1985.
7. A. Chauduri. Some implications of an intensity measure of envy. *Social Choice and Welfare*, 3(4):255–270, 1986.
8. S. Even and A. Paz. A note on cake cutting. *Discrete Applied Mathematics*, 7:285–296, 1984.
9. A. Feldman and A. Kirman. Fairness and Envy. *The American Economic Review*, 64(6):995–1005, 1974.
10. A. Feldman and D. Weiman. Envy, wealth, and class hierarchies. *Journal of Public Economics*, 11(1):81–91, 1979.
11. A. Fink. A note on the fair division problem. *Mathematics Magazine*, 37(5):341–342, 1964.
12. H. Kuhn. On games of fair division. In M. Shubik, editor, *Essays in Mathematical Economics in Honor of Oskar Morgenstern*. Princeton University Press, 1967.
13. C. Lindner and J. Rothe. Degrees of guaranteed envy-freeness in finite bounded cake-cutting protocols. Technical Report cs.GT/0902.0620, ACM Computing Research Repository (CoRR), Sept. 2009. Available on-line at http://arxiv.org/abs/0902.0620.
14. M. Magdon-Ismail, C. Busch, and M. Krishnamoorthy. Cake-cutting is not a piece of cake. In *Proceedings of the 20th Annual Symposium on Theoretical Aspects of Computer Science*, pages 596–607. Springer-Verlag *Lecture Notes in Computer Science #2607*, 2003.
15. A. Procaccia. Thou shalt covet thy neighbor's cake. In *Proceedings of the 21st International Joint Conference on Artificial Intelligence*, pages 239–244. IJCAI, July 2009.
16. J. Robertson and W. Webb. Near exact and envy free cake division. *Ars Combinatoria*, 45:97–108, 1997.
17. J. Robertson and W. Webb. *Cake-Cutting Algorithms: Be Fair If You Can*. A K Peters, 1998.
18. H. Steinhaus. The problem of fair division. *Econometrica*, 16:101–104, 1948.
19. H. Steinhaus. Sur la division pragmatique. *Econometrica*, 17:315–319, 1949. Supplement.
20. H. Steinhaus. *Mathematical Snapshots*. Oxford University Press, New York, 3rd edition, 1969.
21. W. Stromquist. How to cut a cake fairly. *The American Mathematical Monthly*, 87(8):640–644, 1980.
22. A. Tasnádi. A new proportional procedure for the *n*-person cake-cutting problem. *Economics Bulletin*, 4(33):1–3, 2003.
23. G. Woeginger and J. Sgall. On the complexity of cake cutting. *Discrete Optimization*, 4(2):213–220, 2007.