MAGIC: Ergodic Theory Lecture 10 - The ergodic theory of hyperbolic dynamical systems

Charles Walkden

April 12, 2013

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In this lecture we use symbolic dynamics to model more general hyperbolic dynamical systems. We can then use thermodynamic formalism to prove ergodic-theoretic results about such systems.

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$$v_u = \begin{pmatrix} 1 \\ \frac{-1+\sqrt{5}}{2} \end{pmatrix} \qquad v_s = \begin{pmatrix} 1 \\ \frac{-1-\sqrt{5}}{2} \end{pmatrix}.$$

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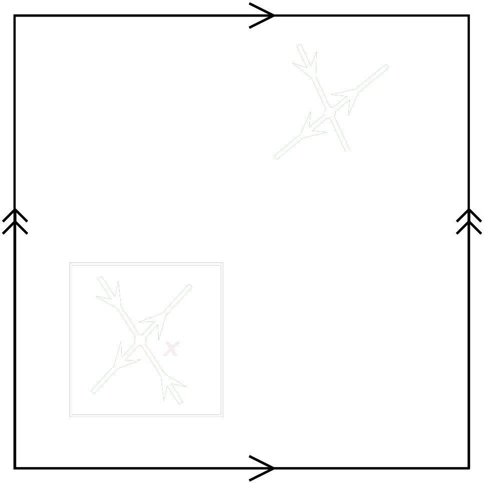

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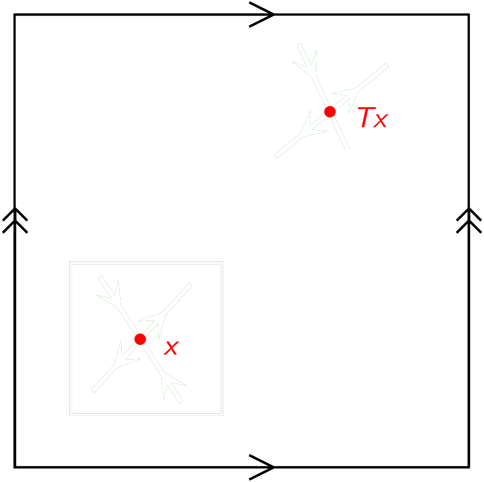

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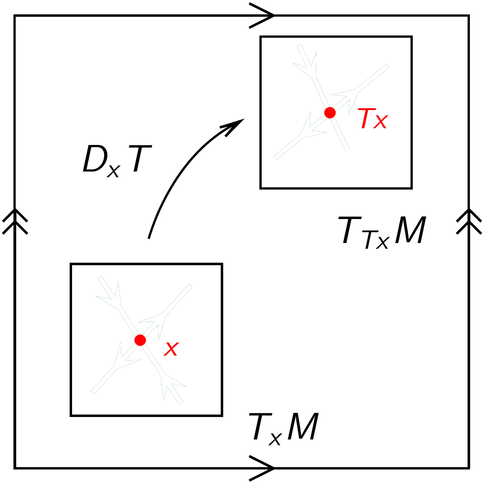

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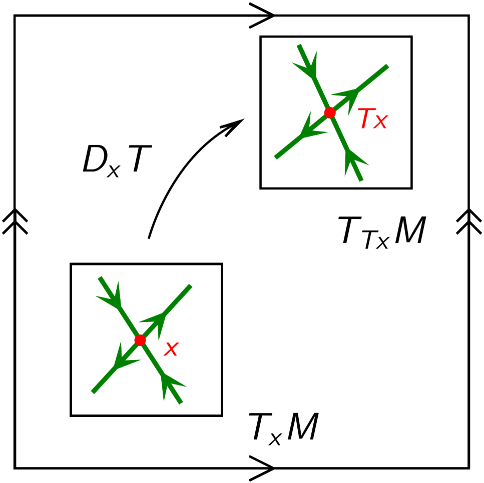

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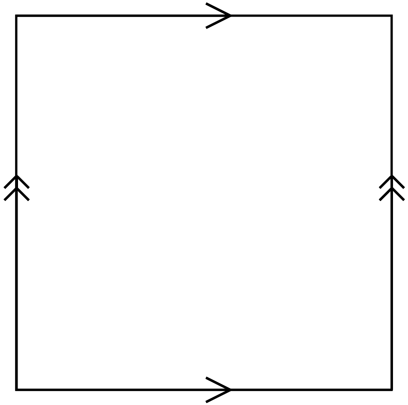

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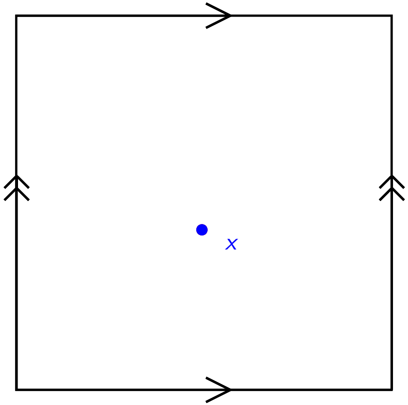

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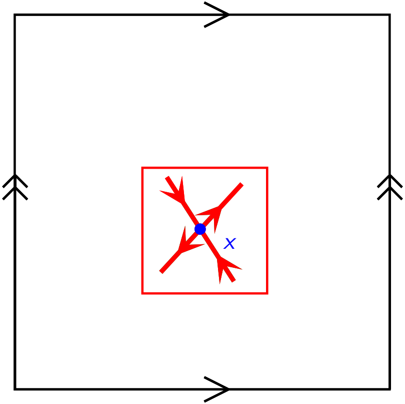

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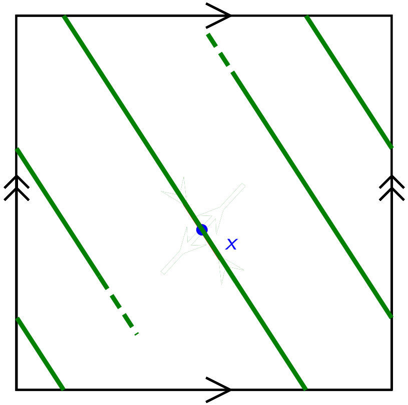

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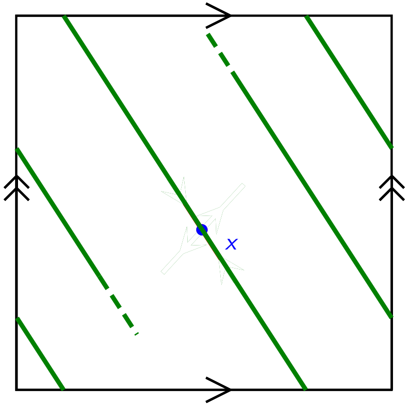

Open Question: Which manifolds support Anosov diffeomorphisms?

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$(x, y, z \in \mathbb{R})$ and let $\Gamma \subset N$ be the cocompact lattice where $x, y, z \in \mathbb{Z}$.

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Topologically, Λ may be very complicated.

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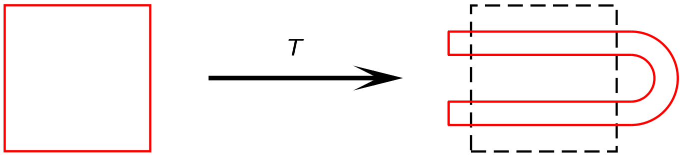

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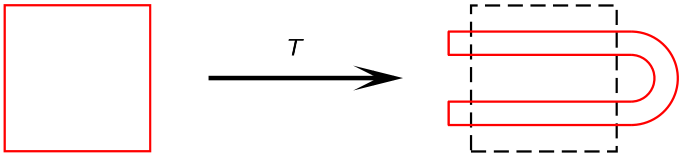
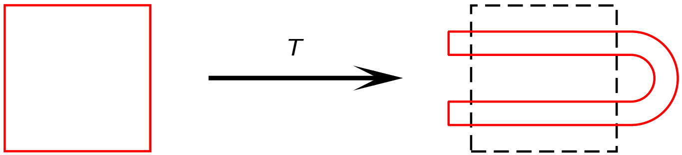


If one iterates T forwards:

Example 1: The Smale horseshoe

Define



If one iterates T forwards:

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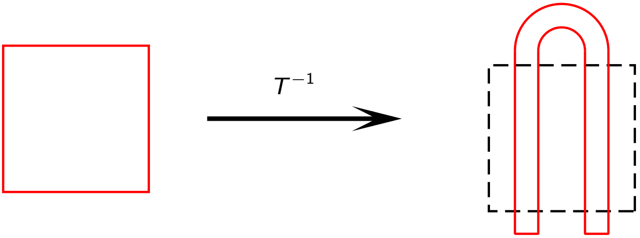

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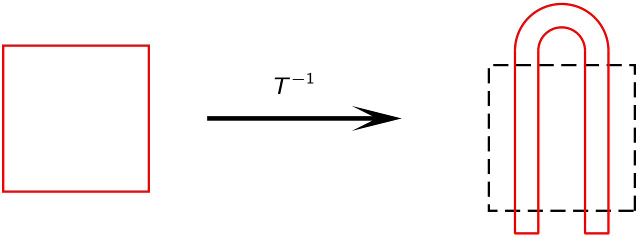

There is a locally maximal hyperbolic set

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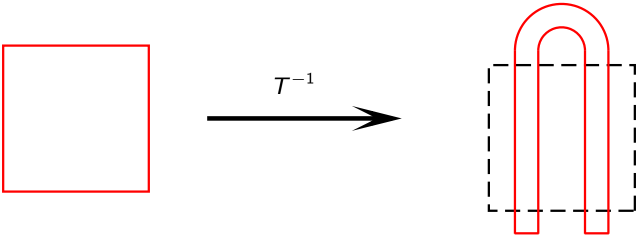

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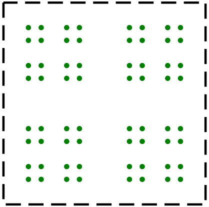

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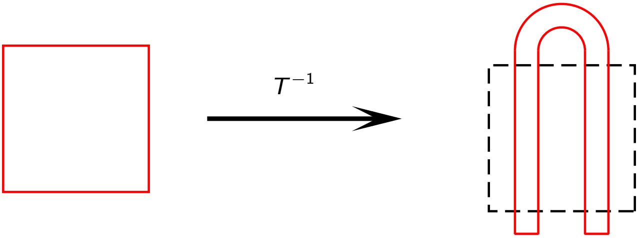

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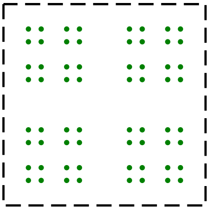

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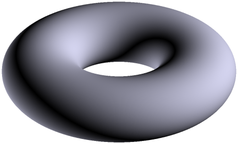

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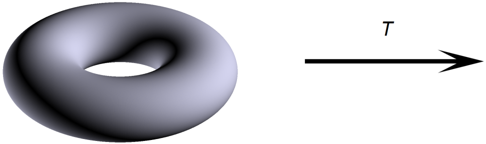

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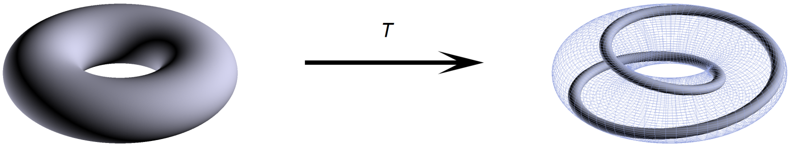

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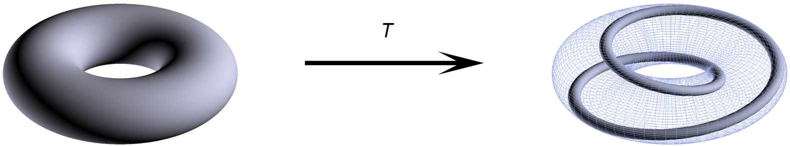

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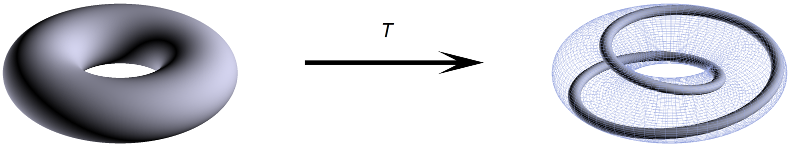

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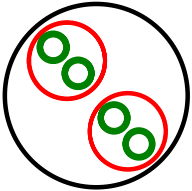

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(Conjecture: Are they all locally the product of a manifold and a Cantor set?)

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(It follows that $d(T^n x, T^n y) \to 0$ exponentially fast as $n \to \pm\infty$ respectively.)

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Thus $[x, y]$ is a point whose orbit approximates that of y (in forward time) and approximates x (in backwards time).

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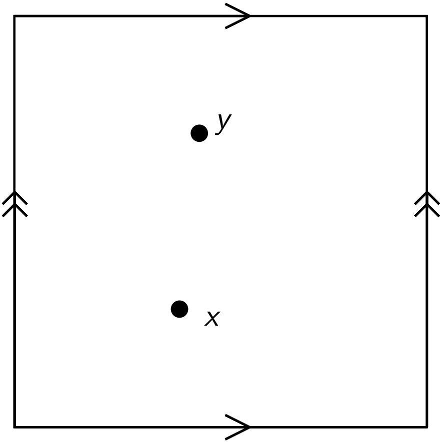

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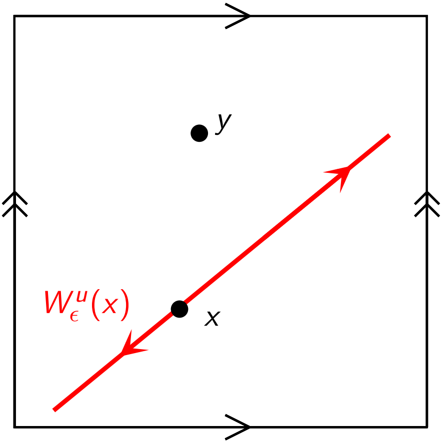

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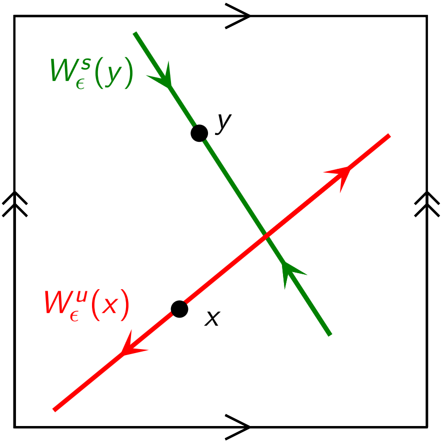

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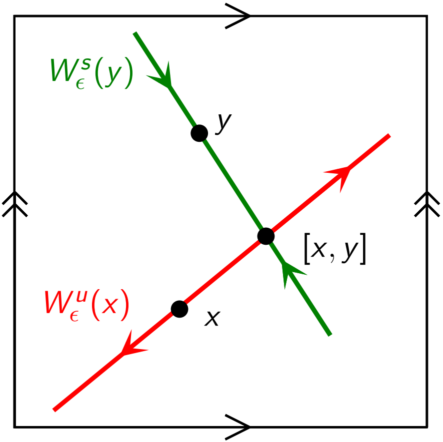

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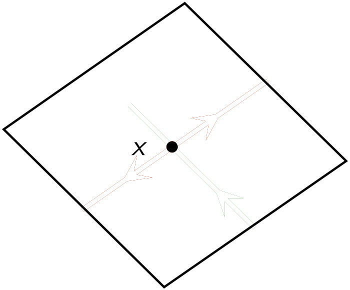

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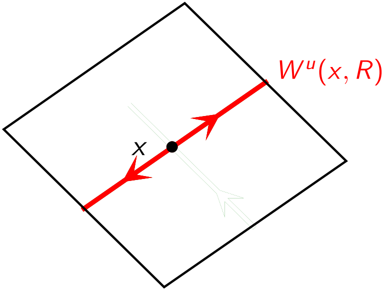

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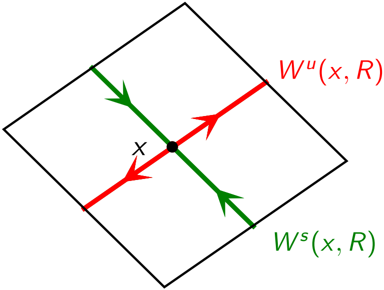

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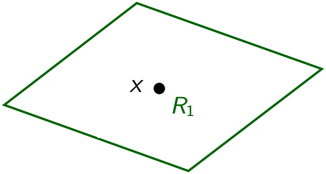

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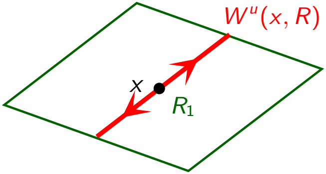

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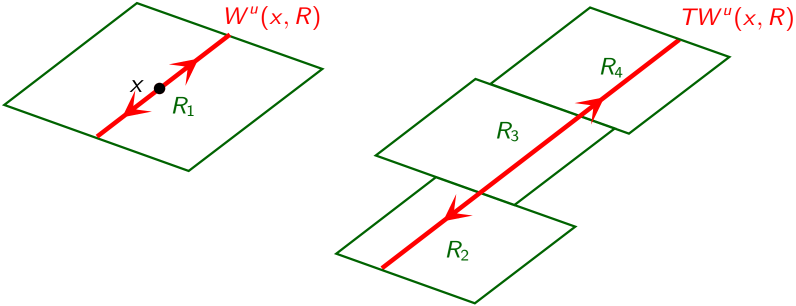

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2. $$\begin{tikzcd} \Sigma \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \Sigma \arrow[d, "\pi"] \\ \Lambda \arrow[r, "T"'] & \Lambda \end{tikzcd}$$ commutes.

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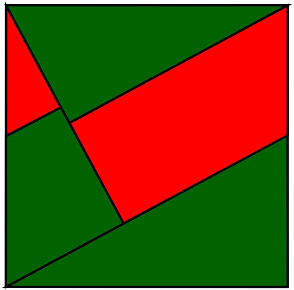

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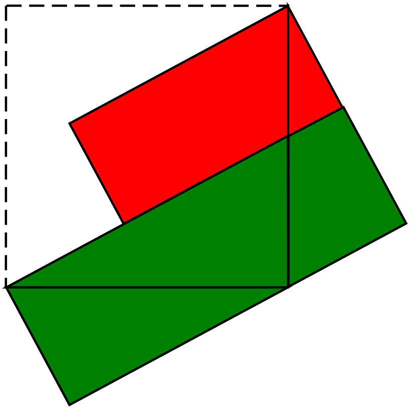

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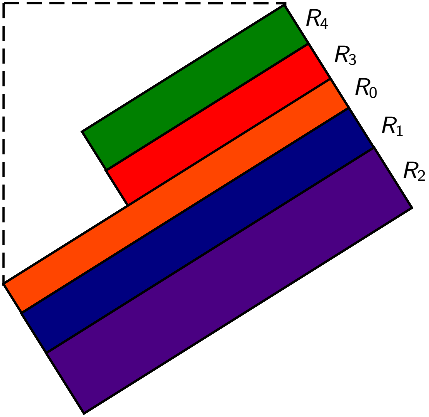

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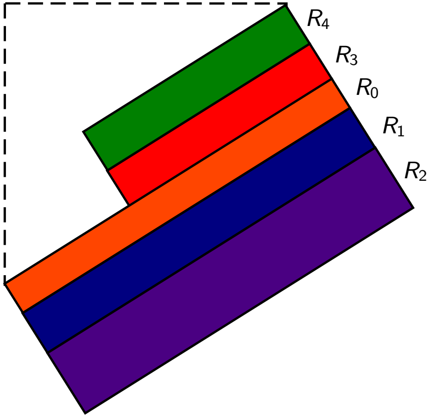

This gives the matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Remark

In general, rectangles may be (geometrically) very complicated. For Anosov automorphisms of a k-dimensional torus, $k \geq 3$, the boundary of a Markov partition will typically be a fractal.

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If f only depends on future coordinates, i.e.

$$f(x) = f(x_0, x_1, \dots)$$

then f can be regarded as being defined on the one-sided shift $f : \Sigma^+ \to \mathbb{R}$.

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Recall: two functions $f, g : \Sigma \to \mathbb{R}$ are cohomologous if $\exists u$ s.t.

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Theorem

Let $f \in F_\theta(\Sigma, \mathbb{R})$. Then f is cohomologous to a function $g \in F_{\theta^{\frac{1}{2}}}(\Sigma^+, \mathbb{R})$ that depends only on the future.

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3. if $f : \Lambda \to \mathbb{R}$ is Hölder then $\hat{f} = f\pi \in F_\theta(\Sigma, \mathbb{R})$ for some $\theta \in (0, 1)$.
4. replace $\hat{f}$ by a cohomologous function $\hat{g} \in F_{\theta^{\frac{1}{2}}}(\Sigma^+, \mathbb{R})$ and apply thermodynamic formalism.

Application 1: Existence of equilibrium states

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Let $\mu_f = \nu_f \circ \pi^{-1}$. Then μ_f is called an equilibrium state for f and is a T-invariant measure.

Application 2: SRB and physical measures

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Let X be a compact Riemannian manifold equipped with the Riemannian volume m.
Let $T : X \to X$ be a smooth diffeomorphism. Typically T does not preserve the volume m. Even if m is T-invariant, then it need not be ergodic.
What can we say about

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x)$$

for m-a.e. $x \in X$?

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(i.e. the set of full measure for which the ergodic sums of continuous observables converges can be chosen to be independent of the observables).

Suppose $T : X \to X$ contains a locally maximal attractor, $T : \Lambda \to \Lambda$ (not necessarily hyperbolic). The *basin of attraction* $B(\Lambda)$ is the set of points that converge under forward iteration to Λ.

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Idea: We think of m-a.e. point as being 'typical', in the sense that m is a naturally occurring measure.

Question: What happens to ergodic sums of continuous observables for m-a.e. point?
i.e. does $\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x)$ exist for all continuous f, m-a.e., and what is the limit?

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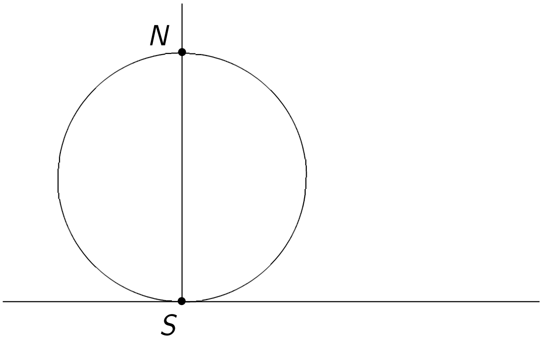

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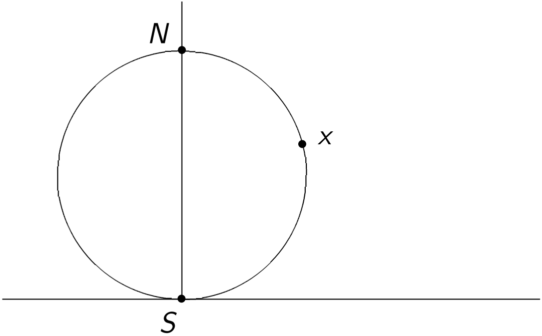

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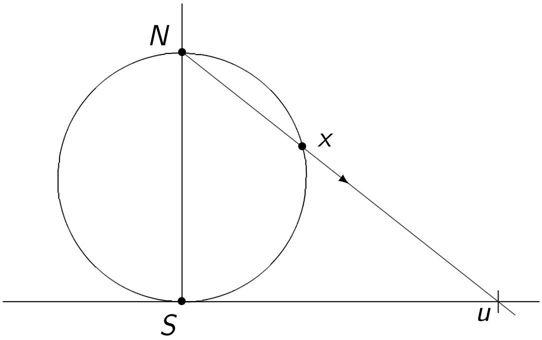

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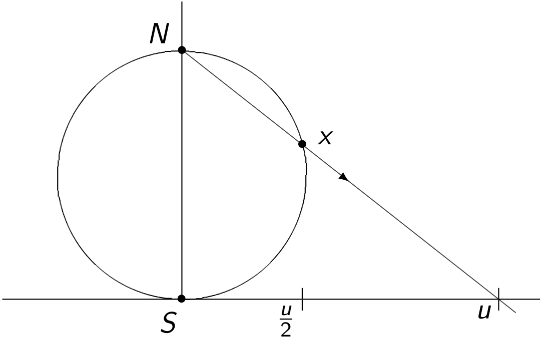

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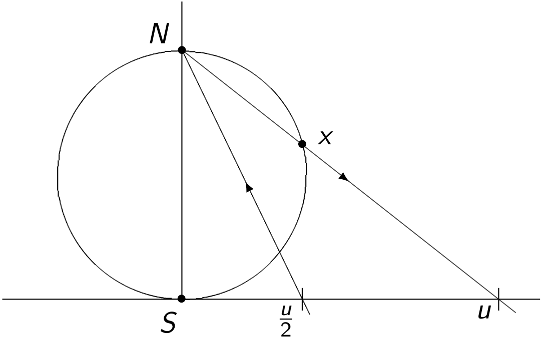

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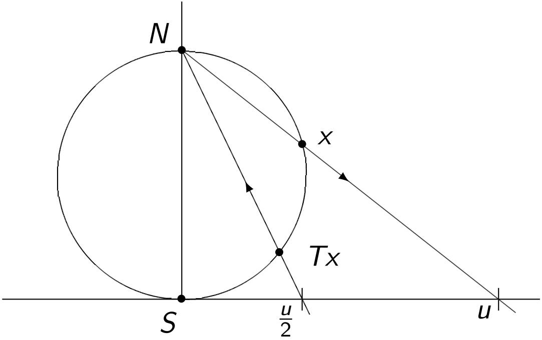

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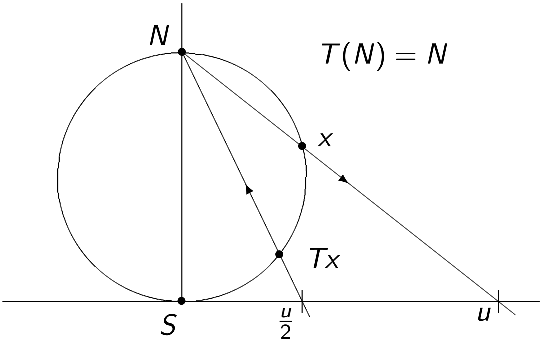

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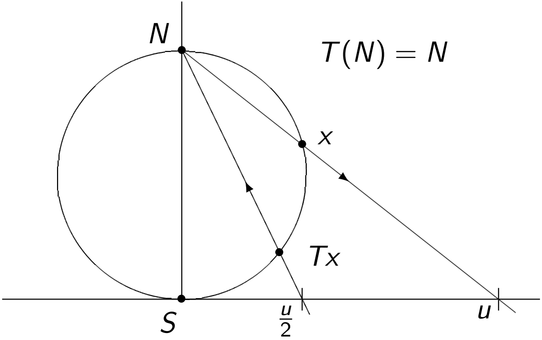

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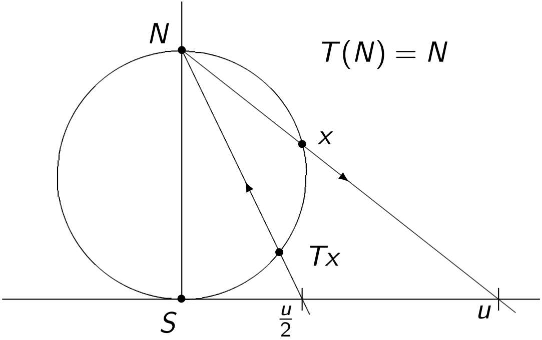

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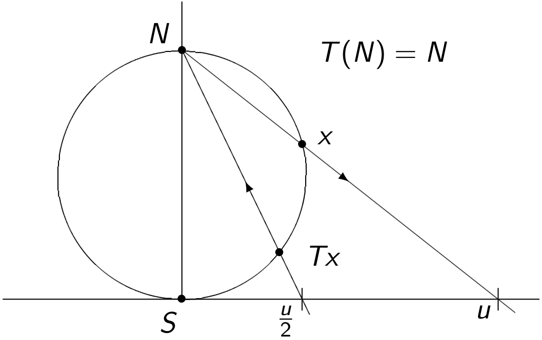

Then S is an attractor with basin $S^1 \setminus \{N\}$. Let $f \in C(X, \mathbb{R})$. As $T^n x \to S \; \forall x \in S^1 \setminus \{N\}$, we have

$$\frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) \longrightarrow f(S) = \int f \, d\delta_S.$$

Definition

Let $T : \Lambda \to \Lambda$ be an attractor. A T-invariant probability measure μ is an *SRB (Sinai-Ruelle-Bowen) measure* if

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The SRB measure is supported on the attractor Λ.
As Λ may be (topologically) small, it may have zero Riemannian volume. (Example: the solenoid has zero volume.) Hence the SRB measure may be very different to the volume.

Theorem

Let $T : \Lambda \to \Lambda$ be a $C^{1+\alpha}$ hyperbolic attractor. Then there is a unique SRB measure. Moreover, it corresponds to the invariant Gibbs measure with potential $-\log dT|_{E^u}$

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Remark

Suppose T is an Anosov diffeomorphism and preserves volume. (Example, the cat map preserves Lebesgue measure = volume.) Then volume is the SRB measure. For a generic Anosov diffeomorphism, the SRB measure is not equal to volume.

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