MAGIC: Ergodic Theory Lecture 9 - Thermodynamic Formalism

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Introduction

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This is the so-called "thermodynamic formalism", which we introduce in this lecture. We will not study the connections between ergodic theory and statistical mechanics; instead we set things up in a way that allows us to study hyperbolic dynamics.

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Define the shift map

$$\sigma(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots).$$

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Define $\|f\|_\theta = |f|_\infty + |f|_\theta$. Then $\| \cdot \|_\theta$ is a norm on the Banach space

$$F_\theta(\mathbb{R}) = \{f : \Sigma \to \mathbb{R} \mid \|f\|_\theta < \infty\}.$$

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In particular, if f is locally constant then f is continuous. (The zero-dimensionality of Σ guarantees the existence of lots of locally constant functions.)

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Example: Take Σ =full one-sided 2-shift. Take $f(x) \equiv \log 1/2$. Then

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We are interested in the spectral properties of L_f.

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Let f be the weight function depending on 2 co-ordinates:

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Consider the action of L_f on functions that only depend on 1 co-ordinate $w(x) = w(x_0)$.

$$\begin{aligned} L_f w(0) &= e^{f(00)} w(0) + e^{f(10)} w(1) = pw(0) + (1-p)w(1) \\ L_f w(1) &= e^{f(01)} w(0) + e^{f(11)} w(1) = qw(0) + (1-q)w(1). \end{aligned}$$

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What are the eigenvalues of P?

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In terms of transfer operators: $L_f 1 = 1$ ($1 =$ the function constantly equal to 1). We say that f is *normalised*.

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Note that $P_{i,j} = e^{f(i,j)}$. Hence if $x = (x_0, x_1, \ldots) \in \Sigma$ then

$$\mu_f[x_0, x_1, \ldots, x_{n-1}] = p_{x_0} e^{f(i_0,i_1)} e^{f(i_1,i_2)} \ldots e^{f(i_{n-2},i_{n-1})} = p_{x_0} e^{f^n(x)}$$

where $f^n(x) = \sum_{j=0}^{n-1} f(\sigma^j x)$.

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What if we allow non-locally constant functions f?

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- The probability measure $d\mu_f = \frac{1}{\int h \, d\nu} h \, d\nu$ is a σ-invariant measure.

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Let Σ be a shift of finite type defined by an aperiodic matrix. Let $f \in F_\theta(\mathbb{R})$ be a weight function. Define $L_f : F_\theta(\mathbb{C}) \to F_\theta(\mathbb{C})$ by

$$L_f w(x) = \sum_{y \text{ s.t. } \sigma(y)=x} e^{f(y)} w(y).$$

- $\exists$ a simple maximal positive eigenvalue $\lambda > 0$ of L_f.
- $\exists$ a strictly positive eigenfunction $0 < h \in F_\theta(\mathbb{R})$ s.t. $L_f h = \lambda h$.
- The remainder of the spectrum of L_f is contained within a disc of radius $< \lambda$.
- $\exists$ a unique probability measure ν s.t. $L_f^* \nu = \lambda \nu$ (i.e. $\int L_f w \, d\nu = \lambda \int w \, d\mu$.
- The probability measure $d\mu_f = \frac{1}{\int h \, d\nu} h \, d\nu$ is a σ-invariant measure.

We call μ_f the *equilibrium state* of f.

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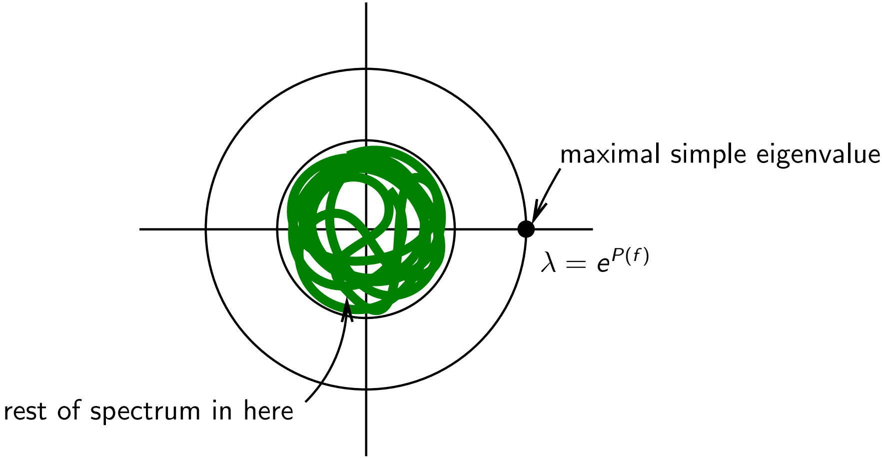
The spectrum of L_f acting on F_θ looks like:



Gibbs measures

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A probability measure (not nec. invariant) m is a *Gibbs measure* if there are constants $A, B > 0$, $C \in \mathbb{R}$ s.t. for all $x \in \Sigma$

$$A \leq \frac{m[x_0, \ldots, n_{n-1}]}{e^{f^n(x) - nC}} \leq B$$

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- Not all constants are equal. (Orwellian equal, rather than numerically equal!) The values of A, B don't matter. C is the pressure of f.
- Given a potential f, is there an *invariant* Gibbs measure? Is it unique?

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Let $f \in F_\theta(\mathbb{R})$. By Ruelle's Perron-Frobenius theorem, L_f has a maximal eigenvalue at $e^{P(f)}$. There is a maximal eigenfunction h s.t. $L_f h = e^{P(f)} h$ and a maximal eigenmeasure ν s.t. $L_f^* \nu = e^{P(f)} \nu$. The measure $d\mu_f = \frac{1}{\int h \, d\nu} h \, d\nu$ is a σ-invariant measure.

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Then μ_f is the unique σ-invariant Gibbs measure with potential f:

$$A \leq \frac{\mu_f [x_0, \dots, n_{n-1}]}{e^{f^n(x) - nP(f)}} \leq B.$$

Cohomologous functions

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Two functions $f, g : \Sigma \to \mathbb{R}$ are *cohomologous* if $\exists u \in F_\theta(\mathbb{R})$ s.t.

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Hence, for example,

$$\lim_{n\to\infty} \frac{1}{n} f^n(x) = \lim_{n\to\infty} \frac{1}{n} g^n(x)$$

so the behaviour of cohomologous functions in the ergodic theorem is the same.

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Hence g is normalised.

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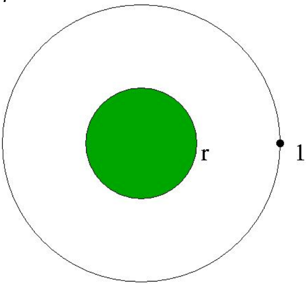

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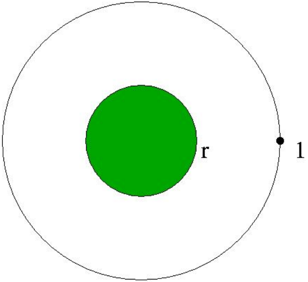

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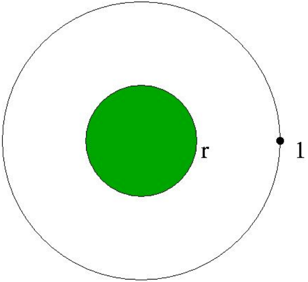

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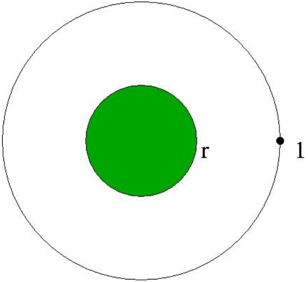

Then $L_f w = \mu(w) + Q(w)$. As eigenprojections are orthogonal:

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Replacing v, w by $v - \int v \, d\mu$, $w - \int w \, d\mu$, it is easy to see that

$$(3) \Leftrightarrow \int v\sigma^n \cdot w \, d\mu \to 0 \;\; \forall v, w \in L^2 \text{ s.t. } \int v \, d\mu = \int w \, d\mu = 0.$$

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We want to show $\int v\sigma^n \cdot w \, d\mu_f \to 0$ for all $v, w \in L^2$, $\int v \, d\mu_f = \int w \, d\mu_f = 0$. It is sufficient to do this for a dense set of functions.

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Remark

In fact, we've shown that equilibrium states (corresponding to Hölder potentials) have a property known as *exponential decay of correlations* (on Hölder functions).

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Another example

Let $\Sigma = \{(x_j)_{j=0}^{\infty} \mid x_j \in \{0,1\}\}$ be the full one-sided 2-shift. Let $f(x) \equiv 0$.

Here $L_f w(x) = \sum_{y \text{ s.t. } \sigma(y)=x} e^{f(y)} w(y) = w(0x) + w(1x)$.

Let $h \equiv 1$. Then $L_f h = 2h$. Hence the maximal eigenvalue for L_f is $\lambda = 2$, i.e. $P(f) = \log 2$.

Let ν be the Bernoulli $(1/2, 1/2)$-measure. It is straightforward to check that $\int L_f w \, d\nu = 2 \int w \, d\nu$, so ν is the eigenmeasure for L_f. Hence we get an invariant measure $d\mu_f = h \, d\nu$. So μ_f is also the Bernoulli $(1/2, 1/2)$-measure.

Recall that μ_f is the measure of maximal entropy for σ:

$$P(f) = \log 2 = h_{\mu_f}(\sigma) = \sup\{h_\mu(\sigma) \mid \mu \text{ is } \sigma\text{-invariant}\}$$

(by the variational principle for entropy).

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- When $f \equiv 0$ this is the variational principle that we had in the last lecture: $P(0)$ is the topological entropy and the corresponding equilibrium state is the measure of maximal entropy.

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- When $f \equiv 0$ this is the variational principle that we had in the last lecture: $P(0)$ is the topological entropy and the corresponding equilibrium state is the measure of maximal entropy.
- We can use equation (4) to *define* pressure for an arbitrary continuous function f and continuous transformation T.

Definitions of pressure

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(Sp $P(0) = h_{top}(\sigma)$ = exponential rate of growth of the number of periodic orbits of period n.)

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  (Suppose $f > 0$. Note that when $s = 0$, $P(-sf) = h_{top}(\sigma) > 0$. Also $P(-sf) \searrow -\infty$ as $s \to \infty$. Hence there is a unique s_0 such that $P(-s_0 f) = 0$. This value of s_0, for particular f, is often of great importance in applications.)

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In the next lecture we discuss the hyperbolic dynamical systems, and see how one can study the ergodic theory of such systems using symbolic dynamics and thermodynamic formalism.