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We also relate entropy to another important quantity: topological entropy.

Throughout: metric entropy $= \text{measure-theoretic entropy} = h_\mu(T)$. 


Recap on entropy

Let $\zeta = \{A_j\}$ be a finite partition of a prob. space $(X, B, \mu)$.

Define the entropy of $\zeta$ $H(\mu)(\zeta) = -\sum_{A \in \zeta} \mu(A) \log \mu(A)$.

If $\zeta, \eta$ are partitions then $\zeta \vee \eta = \{A \cap B | A \in \zeta, B \in \eta\}$.

If $T: X \to X$ is measurable then $T^{-1}\zeta = \{T^{-1}A | A \in \zeta\}$.

The entropy of $T$ relative to $\zeta$ is $h(\mu)(T, \zeta) = \lim_{n \to \infty} \frac{1}{n} H(\mu)\left(\bigcup_{j=0}^{n-1} T^{-j}\alpha\right)$.

The entropy of $T$ is $h(\mu)(T) = \sup\left\{ h(\mu)(T, \zeta) | \zeta \text{ a finite partition} \right\}$.
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$$h_\mu(T, \zeta) = \lim_{n \to \infty} \frac{1}{n} H_\mu \left( \bigvee_{j=0}^{n-1} T^{-j} \alpha \right).$$
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$$h_\mu(T) = \sup \{ h_\mu(T, \zeta) \mid \zeta \text{ a finite partition} \}.$$
Sinai’s theorem

Suppose T is an invertible m.p.t. and \( \zeta \) is a generator. Then 
\[
h_\mu(T) = h_\mu(T, \zeta).
\]

Let \( \sigma \) be the full \( k \)-shift with the Bernoulli \((p_1, \ldots, p_k)\)-measure \( \mu \). Then 
\[
\zeta = \{[1], \ldots, [k]\} \text{ is a generator.}
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\[
h_\mu(\sigma) = h_\mu(\sigma, \zeta) = -k \sum_{j=1}^{k} p_j \log p_j.
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Sinai’s theorem

A finite partition $\zeta$ is a generator if $\bigvee_{j=-n}^n T^{-j} \zeta \nearrow B$. (Equiv. $\bigvee_{j=-n}^n T^{-j} \zeta$ separates $\mu$-a.e. pair of points.)
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The weak* topology

Let $(X, B)$ be a compact metric space with the Borel $\sigma$-algebra. Let $T: X \to X$ be continuous. Let $M(X) = \{\text{all Borel probability measures}\}$. Let $M(X, T) = \{\text{all } T\text{-invariant Borel probability measures}\}$.

A sequence $\mu_n \in M(X)$ weak* converges to $\mu$ ($\mu_n \overset{\ast}{\rightharpoonup} \mu$) if $\int f \, d\mu_n \to \int f \, d\mu$ for all $f \in C(X, \mathbb{R})$.

Q: How does the entropy $h_{\mu}(T)$ vary as a function of $\mu$?
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Q: How does the entropy \(h_\mu(T)\) vary as a function of \(\mu\)?
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Let $\Sigma_2 = \text{full 2-sided 2-shift with shift map } \sigma$. 

Then $h_{\mu_n}(\sigma) = 0$ (as $\mu_n$ is supported on a finite set). 

However, $\mu_n \rightharpoonup \mu$, where $\mu = \text{the Bernoulli (1/2, 1/2)-measure}$. 

Note $h_{\mu}(\sigma) = \log 2$. 
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$x$ is periodic with period $n$ iff
$x = (\cdots x_0x_1 \cdots x_{n-1} x_0x_1 \cdots x_{n-1} \cdots)$. There are $2^n$ points of period $n$. 
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Proof (sketch):

Let \( f = \chi_{[0]} \). Note that

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\int f \, d\mu = \frac{1}{2} \left( \chi_{[0]}(\ldots00\ldots) + \chi_{[0]}(\ldots01\ldots) + \chi_{[0]}(\ldots10\ldots) + \chi_{[0]}(\ldots11\ldots) \right) = \frac{1}{2} = \mu(\chi_{[0]}) = \int f \, d\mu.
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In general,

\[
\int \chi_{[i_0,\ldots,i_{m-1}]} \, d\mu_n = \int \chi_{[i_0,\ldots,i_{m-1}]} \, d\mu_{n'}
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when \( n \geq m \).

Characteristic functions of intervals are continuous. Finite linear combinations of characteristic functions are dense in \( C(X,\mathbb{R}) \) (by the Stone-Weierstrass theorem). Hence \( \mu_n \rightharpoonup \mu \).

Is the entropy map upper semi-continuous? i.e. does \( \mu_n \rightharpoonup \mu \Rightarrow \limsup_{n \to \infty} h_{\mu_n}(T) \leq h_{\mu}(T) \)?

Answer: no in general, yes in many important cases.
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Expansive homeomorphisms

Definition
A homeomorphism $T$ is expansive if:

$\exists \delta > 0$ s.t. if $d(T^n x, T^n y) \leq \delta$ for all $n \in \mathbb{Z}$ then $x = y$.

Example
A shift of finite type is expansive.

Recall $d(x, y) = \frac{1}{2^n}$, $n =$ first disagreement. Let $\delta < 1$. If $x_n \neq y_n$ then $d(T^n x, T^n y) = \frac{1}{2^n} \geq \delta$.

Example
Let $T: \mathbb{R}^k/\mathbb{Z}^k \to \mathbb{R}^k/\mathbb{Z}^k$, $T x = A x \mod 1$ be a toral automorphism given by $A \in \text{SL}(2, \mathbb{R})$. Then $T$ is expansive iff $A$ is hyperbolic (no eigenvalues of modulus 1).

Other examples: all Anosov diffeomorphisms, Smale horseshoe, solenoid,...
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Theorem

Let $T$ be an expansive homeomorphism of a compact metric space. Then the entropy map is upper semi-continuous: if $\mu_n, \mu \in \mathcal{M}(X, T)$, $\mu_n \rightharpoonup \mu$ then

$$\limsup h_{\mu_n}(T) \leq h_{\mu}(T).$$

Proof (sketch):

Fact: Suppose $\mu_n \rightharpoonup \mu$. If $B \in B$ is s.t. $\mu(\partial B) = 0$ then $\mu_n(B) \to \mu(B)$.

If $\zeta$ is a partition such that $\mu(\partial A) = 0 \forall A \in \zeta$ then $H_{\mu_j}(\zeta) \to H_{\mu}(\zeta)$.

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Let $\delta$ be an expansive constant. If $\text{diam} \zeta < \delta$ then $\zeta$ is a generator.

So $h_{\mu}(T) = h_{\mu}(T, \zeta)$ by Sinai.

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Expansive homeomorphisms

Theorem
Let $T$ be an expansive homeomorphism of a compact metric space. Then the entropy map is upper semi-continuous: if $\mu_n, \mu \in M(X, T)$, $\mu_n \rightharpoonup \mu$ then $\limsup h_{\mu_n}(T) \leq h_{\mu}(T)$.

Proof (sketch):

Fact: Suppose $\mu_n \rightharpoonup \mu$. If $B \in \mathcal{B}$ is s.t. $\mu(\partial B) = 0$ then $\mu_n(B) \to \mu(B)$.

If $\zeta$ is a partition such that $\mu(\partial A) = 0 \ \forall A \in \zeta$ then $H_{\mu_j}(\zeta) \to H_{\mu}(\zeta)$. Hence

$$h_{\mu_n}(T, \zeta) \to h_{\mu}(T, \zeta).$$

Let $\delta$ be an expansive constant. If $\operatorname{diam} \zeta < \delta$ then $\zeta$ is a generator. So $h_{\mu}(T) = h_{\mu}(T, \zeta)$ by Sinai. Alter $\zeta$ slightly to ensure $\mu(\partial A) = 0 \ \forall A \in \zeta$. 
Topological entropy

Let $X$ be compact metric, let $T: X \to X$ be continuous. Recall $X$ compact $\Rightarrow$ every open cover of $X$ has a finite subcover.

Definition

Let $\alpha$ be an open cover of $X$. Let $N(\alpha) < \infty$ be the cardinality of the smallest finite subcover of $X$. Define the entropy of $\alpha$ to be $H_{\text{top}}(\alpha) = \log N(\alpha)$.

Definition

Let $\alpha = \{A_i\}$, $\beta = \{B_j\}$ be open covers. The join is the open cover $\alpha \vee \beta = \{A_i \cap B_j | A_i \in \alpha, B_j \in \beta\}$.

Definition

We say $\alpha \leq \beta$ if every element of $\beta$ is a subset of an element of $\alpha$. (Example: $\alpha \leq \alpha \vee \beta$.) Easy check: $\alpha \leq \beta \Rightarrow H_{\text{top}}(\alpha) \leq H_{\text{top}}(\beta)$.

Definition

$T^{-1}\alpha$ is the open cover $\{T^{-1}A | A \in \alpha\}$. 
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Let $\alpha$ be an open cover of $X$. Let $N(\alpha) < \infty$ be the cardinality of the *smallest* finite subcover of $X$. Define the entropy of $\alpha$ to be

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The topological entropy of $T$ relative to the open cover $\alpha$ is

$$h_{\text{top}}(T, \alpha) = \lim_{n \to \infty} \frac{1}{n} H_{\text{top}} \left( \bigsqcup_{j=0}^{n-1} T^{-j} \alpha \right).$$

Remark

The limit exists as $H_n = H_{\text{top}}(\bigsqcup_{j=0}^{n-1} T^{-j} \alpha)$ is subadditive:

$$H_{n+m} \leq H_n + H_m.$$ 

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$$h_{\text{top}}(T) = \sup \{ h_{\text{top}}(T, \alpha) | \alpha \text{ is an open cover of } X \}.$$
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An alternative definition

Let $X$ be compact metric with metric $d$. Let $T : X \to X$ be continuous. Let $d_n(x, y) = \max_{0 \leq j \leq n-1} d(T^j x, T^j y)$. The balls in this metric are $B_n(x, \epsilon) = \{ y | d(T^j x, T^j y) < \epsilon, 0 \leq j \leq n-1 \}$.

So $x, y$ are $d_n$-close if the first $n$ points in the orbits of $x, y$ are close.

Idea: suppose we can't distinguish two orbits if they are close for the first $n$ iterates. How many such orbits are there?
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Spanning sets

Definition

Let \( n \geq 1 \), \( \varepsilon > 0 \). \( F \subset X \) is \((n, \varepsilon)\)-spanning if the \( d_n \)-balls of radius \( \varepsilon \) and centres in \( F \) covers \( X \):

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X = \bigcup_{x \in F} B_n(x, \varepsilon).
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We want to make spanning sets as small as possible. Let \( p_n(\varepsilon) \) be the cardinality of the smallest \((n, \varepsilon)\)-spanning set.

Let \( p(\varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log p_n(\varepsilon) \).

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Definition
Let $n \geq 1, \varepsilon > 0$. $E \subset X$ is $(n, \varepsilon)$-separated if: $x, y \in E$, $x \neq y$ then $d_n(x, y) > \varepsilon$. 

We want to make separated sets as large as possible. Let $q_n(\varepsilon)$ be the cardinality of the largest $(n, \varepsilon)$-separated set.

Remark
$p_n(\varepsilon) \leq q_n(\varepsilon) \leq p_n(\varepsilon/2)$.

Let $E$ be $(n, \varepsilon)$-separated of cardinality $q_n(\varepsilon)$. Then $E$ is $(n, \varepsilon)$-spanning. Hence $p_n(\varepsilon) \leq q_n(\varepsilon)$. 

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Suppose \( E \) is \((n, \varepsilon)-\)separated of cardinality \( q_n(\varepsilon) \).

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For every \( x \in E \) there exists a \( y \in F \) such that \( x \in B_n(y, \varepsilon/2) \).

This map \( E \rightarrow F : x \mapsto y \) is injective.

(If not, then \( x, x' \in E \) could map to the same \( y \in F \). Then \( d_n(x, x') \leq d_n(x, y) + d_n(y, x) < \varepsilon \). Then \( x = x' \) as \( E \) is \((n, \varepsilon)-\)separated.)

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\[ h(T) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log p_n(\varepsilon) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log q_n(\varepsilon). \]

**Theorem (Bowen)**
The definition of topological entropy using open sets agrees with the definition of topological entropy using spanning/separated sets.

**Proof (sketch):**
Careful analysis using Lebesgue numbers of open covers...
Spanning and separated sets

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h_{\text{spanning}}(T) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log p_n(\varepsilon) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log q_n(\varepsilon).
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Calculating topological entropy

Let $\alpha = \{A_1, \ldots, A_k\}$ be a finite open cover. For each $x$, look at the sequence of elements of $\alpha$ the orbit of $x$ visits. This codes the orbit of $x$ by a bi-infinite sequence of symbols from $\{1, \ldots, k\}$. This coding may not be 'nice': different points may have the same coding, the coding may not be unique, the set of all sequences may be complicated (eg: not of finite type).

$\alpha$ is a (topological) generator if each sequence codes at most one point. Precisely, $\alpha$ is a generator if for each sequence $(i_j)_{\infty}^{\infty}$, $\bigcap_{j=\infty}^{-\infty} T^{-j} A_i = \emptyset$ or $1$.
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$$\text{card} \bigcap_{j=-\infty}^{\infty} T^{-j}A_{i_j} = 0 \text{ or } 1.$$
Calculating topological entropy

Proposition

T has a (topological) generator iff T is expansive.

Proof (sketch):
Suppose T is expansive with expansive constant δ. Consider the open cover by balls of radius δ/2. Let α be a finite subcover. Then α is a (topological) generator.

The converse is slightly more involved.
Proposition

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Calculating topological entropy using generators

Proposition

Let $T$ be an expansive homeomorphism and let $\alpha$ be a generator. Then $h_{\text{top}}(T) = h_{\text{top}}(T, \alpha)$.

Proof (sketch):

Step 1:

Clearly $h_{\text{top}}(T, \alpha) \leq h_{\text{top}}(T)$.

Step 2:

$d(\bigsqcup_{n,j} T^{-n} \alpha) \to 0$.

(If $d(\bigsqcup_{n,j} T^{-n} \alpha) \to \epsilon > 0$ then two points could have the same coding - contradicting $\alpha$ being a generator.)

Step 3:

Let $\beta$ be any open cover. Let $r > 0$ be a Lebesgue number for $\beta$.

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Then $h_{\text{top}}(T, \beta) \leq h_{\text{top}}(T, \bigsqcup_{n,j} T^{-n} \alpha) = h_{\text{top}}(T, \alpha)$.

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$h_{\text{top}}(T, \beta) \leq h_{\text{top}} \left(T, \bigvee_{j=-n}^{n} T^{-j}\alpha \right) = h_{\text{top}}(T, \alpha)$. Take the supremum over all $\beta$. 
Let $\sigma: \Sigma^k \to \Sigma^k$ be the full two-sided $k$-shift. Let $\alpha = \{[1], \ldots, [k]\}$. Note $\alpha$ is an open cover of $\Sigma^k$. It's clear that $\alpha$ is a (top.) generator.

Note $\bigvee_{n-1}^{j=0} \sigma^{-j} \alpha$ is the open cover of $\Sigma^k$ into all cylinders of length $n$. There are $k^n$ of these and all of them are needed to cover $\Sigma^k$. Hence $h_{\text{top}}(\sigma) = h_{\text{top}}(\sigma, \alpha) = \lim_{n \to \infty} \frac{1}{n} \log k^n = \log k$. 
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Let $A$ be an irreducible $k \times k$ 0–1 matrix. Let $\sigma : \Sigma_A \to \Sigma_A$ be the shift of finite type. Let $\alpha = \{[1], \ldots, [k]\}$. Again, this open cover is a generator.
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where $\lambda > 0$ is the largest eigenvalue of $A$, by the spectral radius formula.
The variational principle

We can relate metric and topological entropy

Theorem (The variational principle)

Let \( T \) be a continuous transformation of a compact metric \( X \).

Then

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\begin{align*}
\mathcal{H}_{\text{top}}(T) &= \sup \{ \mathcal{H}_{\mu}(T) | \mu \in \mathcal{M}(X, T) \}.
\end{align*}
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Remark

There are examples to show that this supremum need not be achieved.

Definition

Let

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\mathcal{M}_{\text{max}}(X, T) = \{ \mu \in \mathcal{M}(X, T) | \mathcal{H}_{\text{top}}(T) = \mathcal{H}_{\mu}(T) \}
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denote the set of all measures of maximal entropy.
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Proposition
If the entropy map is upper semi-continuous then $\mathcal{M}_{\text{max}}(X, T) \neq \emptyset$.

Proof: an upper semi-continuous function on a compact metric space achieves its supremum. Note that $\mathcal{M}(X, T)$ is a compact metric space.

Remark
Hence expansive homeomorphisms always have at least one measure of maximal entropy. In many cases, there is a unique measure of maximal entropy.
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Hence expansive homeomorphisms always have at least one measure of maximal entropy.

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Let $\sigma: \Sigma_k \to \Sigma_k$ be the full two-sided $k$-shift. Then the Bernoulli $(1/k, \ldots, 1/k)$-measure is the unique measure of maximal entropy.

Proof.

We know the topological entropy of $\sigma$ is $\log k$.

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We show: if $\mu \in M(X, T)$ has $h_\mu(\sigma) = \log k$ then $\mu$ is the Bernoulli $(1/k, \ldots, 1/k)$-measure.

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We need the following fact:

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If $\eta = \{A_1, \ldots, A_\ell\}$ is a finite partition then

$$H(\eta) = \sum_{i=1}^{\ell} \mu(A_i) \log A_i \leq \log \ell$$

with equality iff $\mu(A_i) = 1/\ell$, $1 \leq i \leq \ell$.

(This follows from concavity of $-\log t$.)

So

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(This follows from concavity of $-t \log t$.) So

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Hence $H_\mu \left( \bigvee_{j=0}^{n-1} \sigma^{-j} \alpha \right) = \log k^n$. 

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Hence $\mu$ assigns measure $1/k^n$ to each cylinder.
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Hence \( H_\mu \left( \bigvee_{j=0}^{n-1} \sigma^{-j} \alpha \right) = \log k^n \). So by the fact, each element of \( \bigvee_{j=0}^{n-1} \sigma^{-j} \alpha \) has the same measure \( 1/k^n \). Hence \( \mu \) assigns measure \( 1/k^n \) to each cylinder. So \( \mu \) and the Bernoulli \((1/k, \ldots, 1/k)\)-measure agree on cylinders. By the Kolmogorov Extension Theorem, \( \mu \) is the Bernoulli \((1/k, \ldots, 1/k)\)-measure.
The Parry measure

Let $A$ be an irreducible $0-1$ matrix with corresponding shift of finite type $\Sigma_A$. We show how to construct the measure of maximal entropy.

Theorem (Perron-Frobenius)

Let $A$ be a non-negative irreducible matrix. Then

- there is a positive maximal eigenvalue $\lambda > 0$ s.t. all other eigenvalues satisfy $|\lambda_j| < \lambda$,
- moreover $\lambda$ is simple;
- there are positive left- and right-eigenvectors $u = (u_1, \ldots, u_k)$, $v = (v_1, \ldots, v_k)^T$, $\sum_i u_i = \sum_j v_j = 1$, s.t. $uA = \lambda u$, $Av = \lambda v$.

Apply Perron-Frobenius to $A$ and define $P_{i,j} = A_{i,j}v_j \lambda v_i$, $p_i = u_i v_i c$, where $c = \sum_j u_j v_j$.

Then $P$ is stochastic and $pP = p$.

We define the Parry measure to be the Markov measure $\mu_{[i_0, i_1, \ldots, i_n]} = p_{i_0} P_{i_0, i_1} \cdots P_{i_{n-1}, i_n}$.
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Recall that for a Markov measure $\mu$ given by the stochastic matrix $P$, we have

$$h_\mu(\sigma) = -\sum_{i,j} p_{i,j} \log p_{i,j}.$$ 

It's an easy check that the Parry measure $\mu$ has

$$h_\mu(\sigma) = \log \lambda.$$ 

We already know that the topological entropy of $\sigma$ is $\log \lambda$. Hence the Parry measure is a measure of maximal entropy.

Proposition

Let $A$ be an irreducible $0-1$ matrix with corresponding shift of finite type $\Sigma_A$. Then the Parry measure is the unique measure of maximal entropy.
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Let $A$ be an irreducible $0^{-1}$ matrix with corresponding shift of finite type $\Sigma_A$. Then the Parry measure is the unique measure of maximal entropy.
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Let $A$ be an irreducible $0 – 1$ matrix with corresponding shift of finite type $\Sigma_A$. Then the Parry measure is the unique measure of maximal entropy.
Towards thermodynamic formalism

Many other dynamical systems have measures of maximal entropy. Lebesgue measure is the unique measure of maximal entropy for a linear hyperbolic toral automorphism.

If the dynamical system $T$ is 'hyperbolic' (in an appropriate sense, but this includes: Anosov diffeomorphisms, Axiom A diffeos on basic sets such as the Smale horseshoe, (in continuous time) geodesic flows on compact negatively curved Riemannian manifolds) then there is a unique measure of maximal entropy. These measures of maximal entropy can often be related to the spectral properties (=maximal eigenvalue) of an associated operator. We will discuss this further in the next lecture.
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