MAGIC: Ergodic Theory Lecture 8 - Topological entropy

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We also relate entropy to another important quantity: topological entropy.

Throughout: metric entropy $=$ measure-theoretic entropy $=$ $h_\mu(T)$.

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$$h_\mu(T) = \sup\{h_\mu(T, \zeta) \mid \zeta \text{ a finite partition}\}.$$

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Let σ be the full k-shift with the Bernoulli $(p_1, \ldots, p_k)$-measure μ. Then $\zeta = \{[1], \ldots, [k]\}$ is a generator.

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Q: How does the entropy $h_\mu(T)$ vary as a function of μ?

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$$\mu_n = \frac{1}{2^n} \sum_{x=\sigma^n x} \delta_x \in M(X, T).$$

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Then $h_{\mu_n}(\sigma) = 0$ (as μ_n is supported on a finite set).

However, $\mu_n \rightharpoonup \mu$, where $\mu =$ the Bernoulli $(1/2, 1/2)$-measure.
Note $h_\mu(\sigma) = \log 2$.

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Answer: no in general, yes in many important cases.

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Other examples: all Anosov diffeomorphisms, Smale horseshoe, solenoid,...

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Idea: suppose we can't distinguish two orbits if they are close for the first n iterates. How many such orbits are there?

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Let $n \geq 1, \varepsilon > 0$. $F \subset X$ is (n, ε)-spanning if the d_n-balls of radius ε and centres in F covers X:

$$X = \bigcup_{x \in F} B_n(x, \varepsilon).$$

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Let $n \geq 1, \varepsilon > 0$. $F \subset X$ is (n, ε)-spanning if the d_n-balls of radius ε and centres in F covers X:

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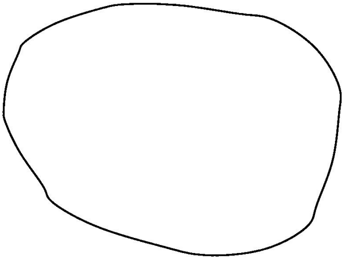

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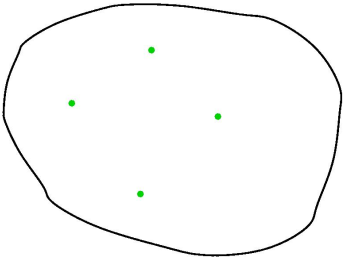

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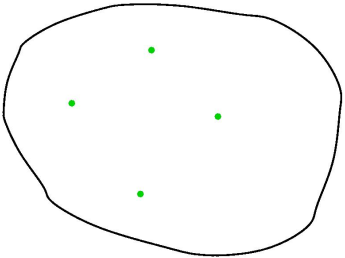

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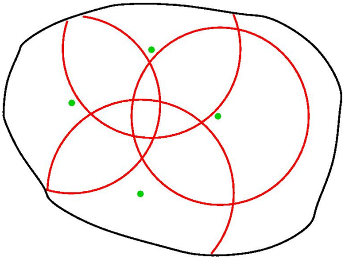

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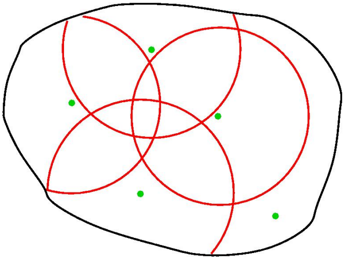

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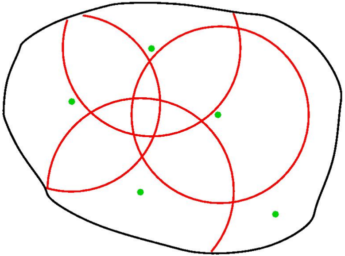

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The definition of topological entropy using open sets agrees with the definition of topological entropy using spanning/separated sets.

Proof (sketch):

Careful analysis using Lebesgue numbers of open covers...

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α is a (topological) generator if each sequence codes at most one point. Precisely, α is a generator if for each sequence $(i_j)_{j=-\infty}^{\infty}$

$$\mathrm{card} \bigcap_{j=-\infty}^{\infty} T^{-j}\overline{A_{i_j}} = 0 \text{ or } 1.$$

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Suppose T is expansive with expansive constant δ. Consider the open cover by balls of radius $\delta/2$. Let α be a finite subcover. Then α is a (topological) generator.

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The converse is slightly more involved.

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where $\lambda > 0$ is the largest eigenvalue of A, by the spectral radius formula.

The variational principle

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Theorem (The variational principle)

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Definition

Let $M_{\max}(X, T) = \{\mu \in M(X, T) \mid h_{\mathrm{top}}(T) = h_\mu(T)\}$ denote the set of all *measures of maximal entropy*.

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Remark

Hence expansive homeomorphisms always have at least one measure of maximal entropy.
In many cases, there is a *unique* measure of maximal entropy.

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Let $\sigma : \Sigma_k \to \Sigma_k$ *be the full two-sided* k*-shift. Then the Bernoulli* $(1/k, \ldots, 1/k)$*-measure is the unique measure of maximal entropy.*

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Let $\zeta = \{[1], \ldots, [k]\}$. This is a generator, so by Sinai's thm $\log k = h_\mu(\sigma) = h_\mu(\sigma, \zeta)$.

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Note that $h_\mu(\sigma, \zeta) \leq \dfrac{1}{n} H_\mu \left(\bigvee_{j=0}^{n-1} \sigma^{-j} \alpha \right)$ *(as* $H_n = H_\mu \left(\bigvee_{j=0}^{n-1} \sigma^{-j} \alpha \right)$ *is subadditive, so* $\frac{1}{n} H_n \to \inf_n \frac{1}{n} H_n = h_\mu(\sigma, \zeta)$*).*

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Hence μ assigns measure $1/k^n$ to each cylinder. So μ and the Bernoulli $(1/k, \ldots, 1/k)$-measure agree on cylinders. By the Kolmogorov Extension Theorem, μ is the Bernoulli $(1/k, \ldots, 1/k)$-measure.

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Apply Perron-Frobenius to A and define $P_{i,j} = \dfrac{A_{i,j} v_j}{\lambda v_i}$, $p_i = \dfrac{u_i v_i}{c}$, where $c = \sum_{j=1}^{k} u_j v_j$.

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Then P is stochastic and $pP = p$. We define the *Parry measure* to be the Markov measure

$$\mu[i_0, i_1, \ldots, i_n] = p_{i_0} P_{i_0,i_1} \cdots P_{i_{n-1},i_n}.$$

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These measures of maximal entropy can often be related to the spectral properties (=maximal eigenvalue) of an associated operator. We will discuss this further in the next lecture.