# MAGIC: Ergodic Theory Lecture 6 - Continuous transformations of compact metric spaces

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February 28, 2013

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- $(X, \mathscr{B}, \mu)$, a probability space.
- a measure preserving transformation $T : X \longrightarrow X$.

In this lecture, we fix a transformation $T : X \longrightarrow X$ and consider the space $\mathcal{M}(X, T)$ of all $T$-invariant probability measures.

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In order to equip $\mathcal{M}(X, T)$ with some structure, we need some structure on $X$ and $T$. Throughout:

- $X =$ a compact metric space.
- $\mathscr{B} =$ the Borel $\sigma$-algebra (smallest $\sigma$-algebra that contains all open sets).
- $T : X \longrightarrow X$ a continuous transformation.

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Recall:

A space is *separable* if there is a countable dense subset.

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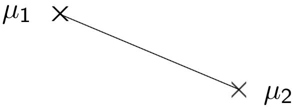

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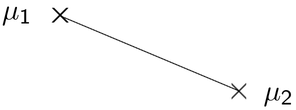


**Proof.**

Immediate from definitions. $\square$

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## Remark

This does not say $\mu_n(B) \longrightarrow \mu(B) \ \forall B \in \mathscr{B}$.

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   Note that $1/n \to 0$ as $n \to \infty$. Let $B = \{0\}$. Then $0 = \delta_{1/n}(B) \not\to \delta_0(B) = 1$.

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The Riesz Representation Theorem says that any functional $C(X, \mathbb{R}) \to \mathbb{R}$ satisfying (1) - (4) is given by integration w.r.t. a suitable Borel probability measure:

## Theorem:(Riesz Representation)

Let $\omega : C(X, \mathbb{R}) \longrightarrow \mathbb{R}$ be a functional satisfying

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Then there exists a *unique* Borel probability measure $\mu \in \mathcal{M}(X)$ s.t. $\omega(f) = \int f\, d\mu$.

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Corollary

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Proof.

Use $\mathcal{M}(X) = C(X, \mathbb{R})^* \cap$ positive cone $\cap$ unit ball. The unit ball in $C(X, \mathbb{R})^*$ is weak-* compact by Alaoglu's theorem. The positive cone on $C(X, \mathbb{R})^*$ is weak-* closed. $\square$

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We want to investigate the structure of $\mathcal{M}(X, T)$. First we need to integrate with respect to $T_*\mu$.

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(easy check that $\chi_{T^{-1}B} = \chi_B \circ T$).

Hence the lemma is true for linear combinations of characteristic functions, hence for positive measurable functions, hence for integrable functions. $\square$

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## Proof.

$\Longrightarrow$:

$$\int f \circ T \, d\mu \underset{\text{Lemma 1}}{=} \int f \, d(T_*\mu) \underset{\text{as } \mu \in \mathcal{M}(X,T)}{=} \int f \, d\mu$$

$\Longleftarrow$: Use Lemma 1 to rewrite (1) as

$$(T_*\mu)(f) = \mu(f) \, \forall f \in C(X, \mathbb{R}).$$

The following gives a useful condition for checking whether a measure is $T$-invariant.

**Lemma 2**

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By uniqueness in the Riesz Representation theorem, $T_*\mu = \mu$. $\square$

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### Theorem (Schauder-Tychonoff fixed point theorem)

*Let $K$ be a compact convex subset of a locally convex space. Let $T : K \to K$ be continuous. Then $T$ has a fixed point in $K$.*

$\mathcal{M}(X)$ is a compact convex subset of the locally convex space $C(X, \mathbb{R})^*$. The map $T_*$ is continuous and maps $\mathcal{M}(X)$ to itself.

Hence there exists $\mu \in \mathcal{M}(X)$ such that $T_*\mu = \mu$, i.e. $\mu \in \mathcal{M}(X, T)$.

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Unravel the definitions! $\square$

# Ergodic measures for continuous transformations

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## Extremal points in convex sets

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*Recall:* $Y$ is *convex* if $\forall y_1, y_2 \in Y, 0 \leq \alpha \leq 1$ we have $\alpha y_1 + (1 - \alpha) y_2 \in Y$.

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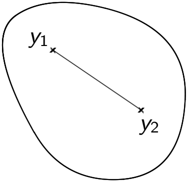

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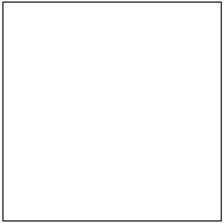
Let $\mathrm{Ext}(Y) = \{\text{extremal points}\}$.

# Examples

$Y =$ unit square



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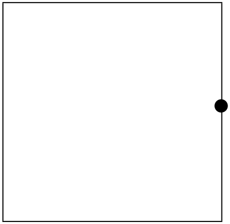

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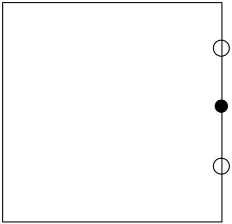

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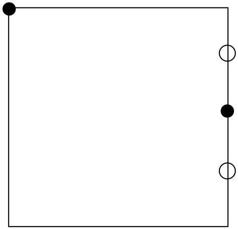

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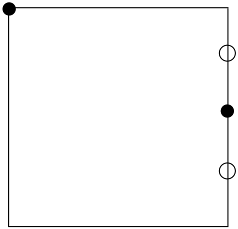

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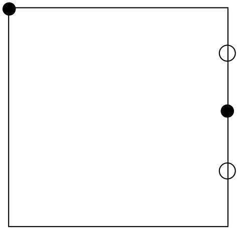

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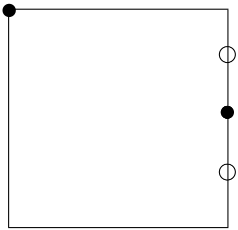

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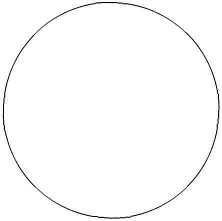

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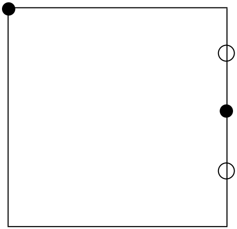

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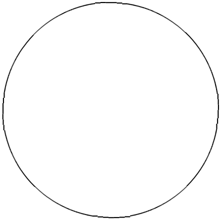

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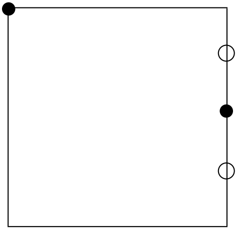

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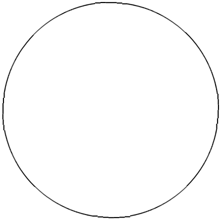

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### Remark

The geometric intuition that the extremal points lie on the boundary fails in infinite dimensions.

### Definition

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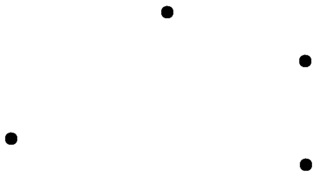

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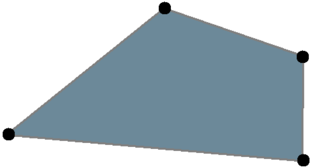

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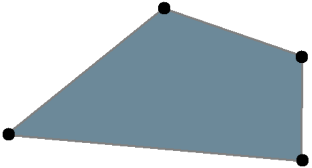

### Theorem:(Krein-Milman)

Let $X$ be a topological vector space on which $X^*$ separates points. Let $K \subset X$ be a compact convex subset.Then $K$ is the convex hull of its extremal points:

$$K = \mathrm{Cov}(\mathrm{Ext}(K)).$$

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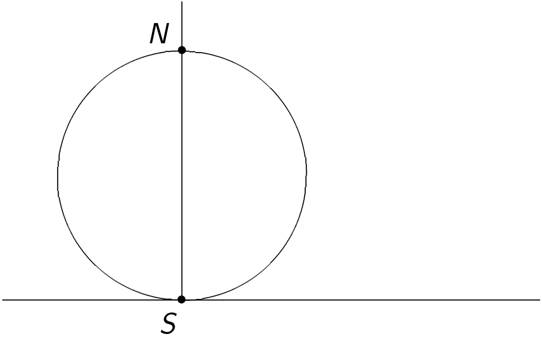

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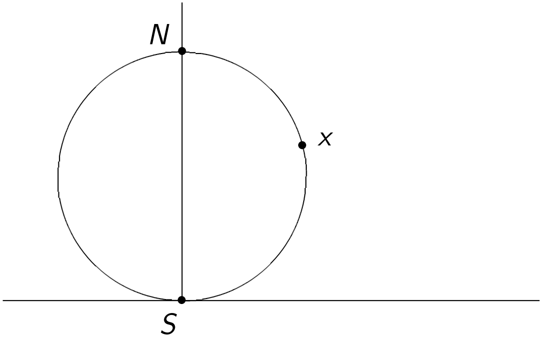

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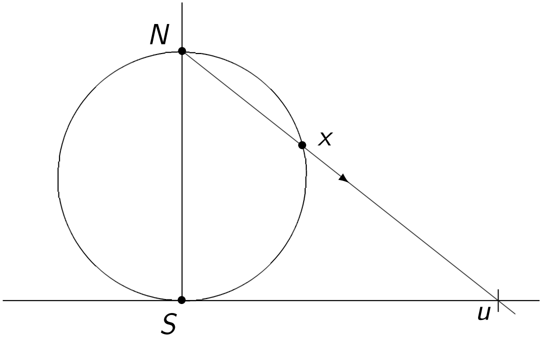

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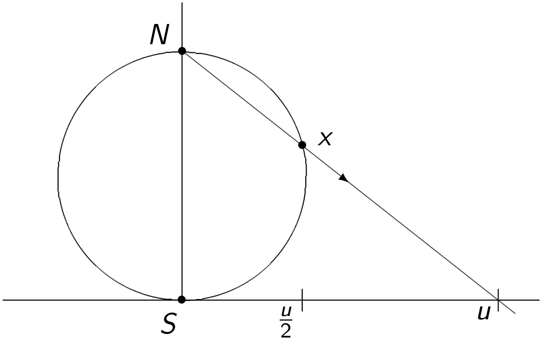

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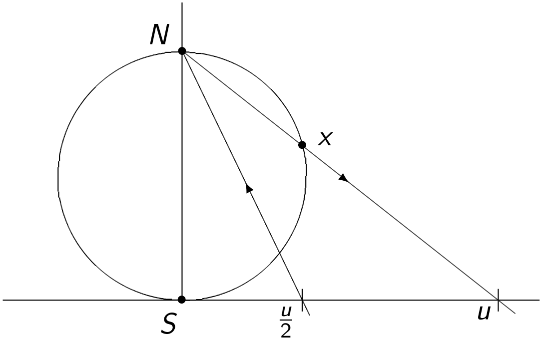

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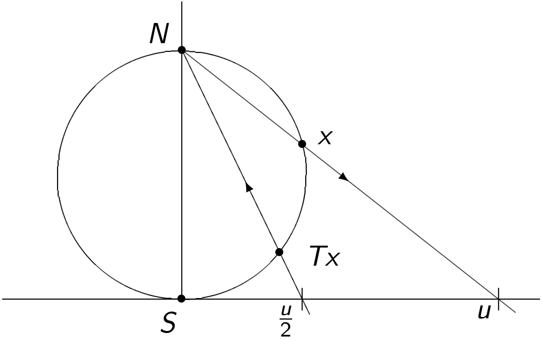

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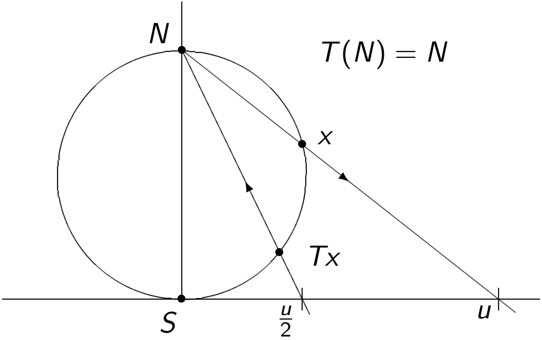

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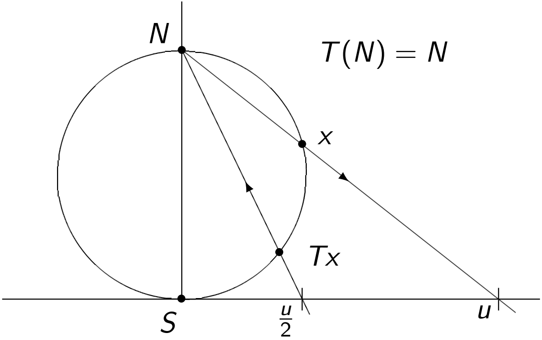


Then $N$, $S$ are fixed points under $T$. Hence $\delta_N, \delta_S \in \mathcal{M}(X, T)$.
We calculate $\mathcal{M}(X, T)$.

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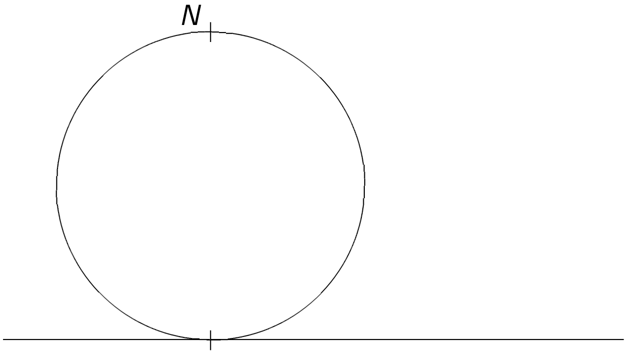

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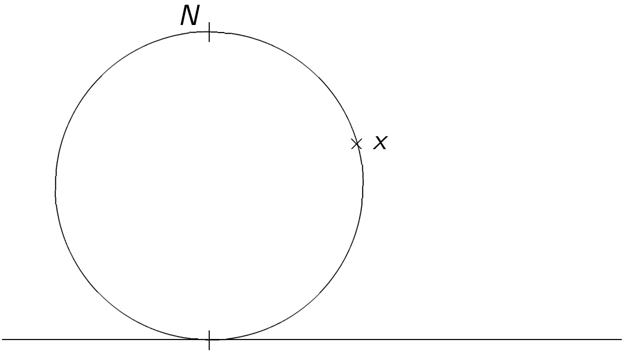

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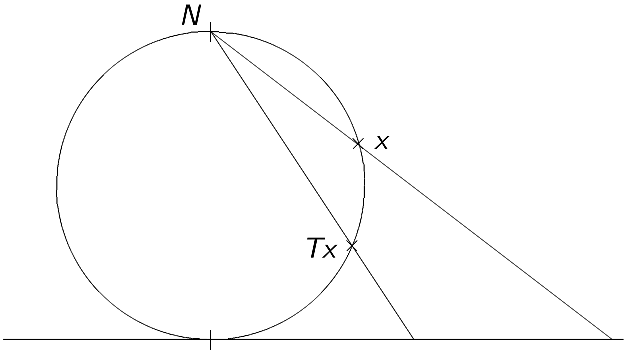

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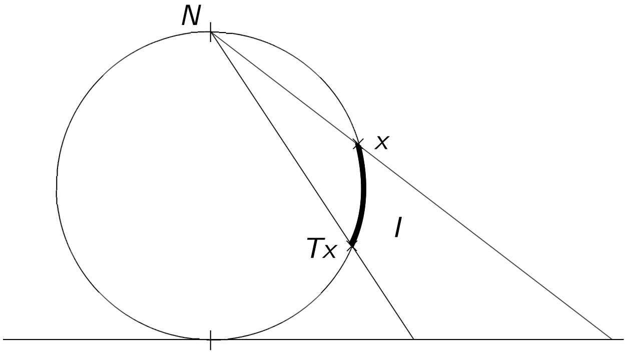

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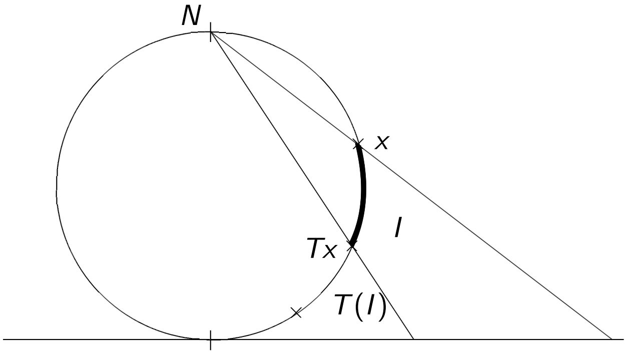

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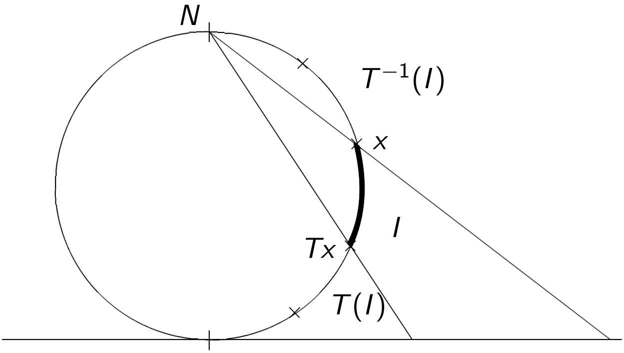

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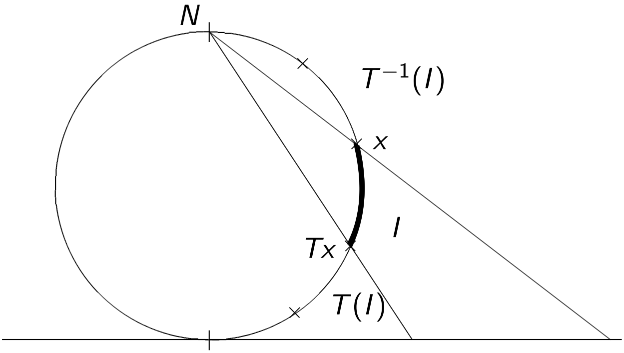


Then the right semicircle is the disjoint union $\bigcup_{n=-\infty}^{\infty} T^{-n}I$.

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This has extremal points $\delta_N, \delta_S$, which are precisely the ergodic measures.

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### Example

Let $T$ be the doubling map $X \longrightarrow X$. Then $\mathcal{M}(X, T)$ is infinite dimensional. The ergodic measures are precisely the extremal points of $\mathcal{M}(X, T)$. However, the set of ergodic measures is also weak-* dense in $\mathcal{M}(X, T)$.

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Unique ergodicity implies the following strong convergence result.

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Remark

Thus unique ergodicity holds iff we have uniform convergence $\forall x \in X$ in the ergodic theorem (for continuous observables).

## Example

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By the Riesz Representation theorem, $m = \mu$. $\square$

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We show how to calculate entropy for a number of examples.